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Effect of Resistive Force and Earth's Ellipticity on Resonant Curves of Geo-centric Satellite Using Unperturbed Solution

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Abstract

This research paper examines the impact of the resistive force and equatorial ellipticity of the Earth (EEE) on the motion of a geocentric satellite. We express a satellite's motion in a spherical coordinate system using potential of the Earth. We apply an unperturbed solution to simplify and reduce the equations into an ODE of second-order. After that, we analyze the resonant curves and oscillatory amplitudes using the differential equation's particular solution. We observe that the resonance arises for the frequencies $\dot{\vartheta}_0$ (satellite's angular velocity) and $\dot{\gamma}$ (rate of change of EEE). Further, we analyze motion of the satellite in the three different cases: (i) $\phi = 0$ and $b = 0$, (ii) $\phi = 0$ and b 6= 0, and (iii) ϕ = 06 and b 6= 0, with b as a coefficient of resistive force and ϕ as a latitude of satellite. Later, we examine the effect of γ, b, and orbital elements on each case's resonant curves and oscillatory amplitudes.

Keywords: *Resistive Force, Earth's equatorial ellipticity (EEE), Resonance, Geocentric satellite, Unperturbed solution.*

Introduction

In any dynamic system, resonance plays a fundamental role between more than one frequency, observing the repetition in the geometric configuration of their position in the orbits for a short period. The resonance happens due to the small integer value of nearly zero of the orbital period. Resonance in the satellite's motion occurs due to the effect of tides. Numerous researchers and investigators have studied the resonance in the motion of a satellite in a solar system. However, they have given less attention to the tesseral-harmonic of second-order regarding the EEE and resistive force.

The EEE is an essential aspect of the satellite's motion, defined as the angle measured from the satellite's projection to the minor axis of Earth's equatorial ellipse. Resistive force is the vector sum of numerous forces whose direction is opposite to the body's motion which is proportional to the velocity of a satellite.

Literature review: We have gone through work done by several researchers, and the review of some literature related to our work is given as:

Alimov *et al.*(2001)[1] gave the approximate theory on the satellite's motion for resonant and nonresonant

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points in the spherical coordinate frame. Belyanin and Gurfil (2009)[2] expressed the motion of geostationary satellites. They have examined the effect of a satellite's equinoctial precession and station-keeping requirements on the orbital dynamics. Bhatnagar et al.(1990)[3] analyzed the inplane motion of a satellite for the gravitational force of the Moon, Sun, and oblate Earth, including EEE. Callegari *et al.*(2004)[4] developed a model of two planets near a first-order mean motion resonance for the field of the three-body problem. Donald *et al.*(2013)[5] examined the orbit's dynamics around 3:1 resonance in the Moon system. Elipe *et al.*(2012)[6] studied the problem using the Lyapunov stability method for stationary points towards the central body in the absence of resonances and the in-case of resonances of orders 3 and 4. Frick and Garber (1962)[7] discussed on perturbations of a synchronous Satellite. Marzari *et al.*(2006)[8] studied the stability of planetary orbits near the 2:1 mean motion resonance for a planetary-mass ratio and orbital parameters. Voyatzis *et al.*(2005)[9] analyzed periodic orbits for symmetric and non-symmetric in the resonance's mean motion 1:4, 1:3, and 1:2 for the planar circular R3BP. Vrbik (2013)[10] investigated the resonance in the motion of a test particle of a planar circular R3BP. Yadav *et al.*(2021)[11] analyzed the resonant curves of the geosynchronous satellite using the perturbation technique, including the effect of EEE and resistive force. Yadav and Aggarwal (2013)[12] examined geocentric satellite resonance occurs due to the EEE parameter. Yadav *et al.*(2014)[13] analyzed the motion of a geosynchronous satellite. Also, they have examined the resonance under the gravity of the Sun, the Moon, and the Earth, including EEE. Yadav *et al.*(2022)[14] analyzed the impact of EEE on the resonant curves using an unperturbed solution. Also, they discussed the phase portrait of the geocentric satellite under the gravitational force of the Earth, the Moon, and the Sun.

Motivation of the problem: We have been motivated by the research work done by Yadav and Aggarwal (2013)[12] on the resonance in the motion of a geocentric satellite appearing for the EEE parameter in polar form. With the help of Eath's potential, they have expressed the equation of motion of a satellite as

$$M_s\left(\ddot{r}-r\dot{\vartheta}^2\right)=\frac{\partial U}{\partial r}$$
$$M_s\left(\frac{1}{r}\frac{d(r^2\dot{\vartheta})}{dt}\right)=\frac{1}{r}\frac{\partial U}{\partial \vartheta},$$

They have investigated resonant points for the frequencies, the angular velocity $\vartheta^{\cdot}$ of a geocentric satellite around the Earth, and the rate of change of EEE. Also, the Brown-Shock method examined the amplitude and period of the oscillation at the resonant points. Although, they have not shown the EEE on a particular solution to the problem and did not show the dynamics of resonant curves. We have extended the problem by applying a resistive force to the satellite. Also, we seek the effect of the resistive force coefficient on the resonant curves and orbital elements.

Aim of the paper: In this research work, we investigate the impact of resistive force and EEE on resonant curves of the geocentric satellite. Also, we express the motion's satellite in a spherical coordinate system. Using an unperturbed solution, we reduce the system of equations into an ODE of second-order and obtain the solution in three different cases (i) when the satellite lies on the equatorial plane and the coefficient of resistive force is zero, (ii) when the satellite lies on the equatorial plane and coefficient of resistive force is non-zero, (iii) satellite does not lie on the equatorial plane and coefficient of resistive force is non-zero. Finally, we show the effect of orbital elements and coefficient of resistive on the resonant curves.

The present study of geocentric satellites with resonance has many applications in several fields such as telecommunication, navigation, mass media, meteorology, and many others.

This work presents in; Section 2 provides a statement and configuration of the problem and expresses the satellite's equations of motion in spherical coordinates using the potential of the Earth. In section 3, we carry out the solution in three different cases. Subsection 3.1 provides a solution for the coefficient of resistive force is zero, and the satellite lies on the equatorial plane. Subsection 3.2 examines the resonant points in the satellite's motion from the established solution. Subsection 3.3 establishes a solution when the coefficient of resistive force is non-zero and the satellite lies on the equatorial plane. Also, we analyze the effect of resistive force on resonant curves. In subsection 3.4, we discusses about the motion of satellite when ϕ 6= 0 and b 6= 0. Section 4 discusses the findings and results, and the last section, 5, concludes the results.

Problem statements and equations of motion of a geocentric satellite: In this configuration (Fig.1), it is assumed that the satellite revolves around the center of Earth. Let ~r be a position vector from the center of Earth to the satellite P, and the position of the satellite P is defined in the spherical coordinates r, ϑ, and ϕ. Let (X,Y,Z) be an initial reference coordinate system with the origin at the Earth's center. The angle γ represents the difference between the satellite longitude and the minor axis line, and line OA represents the instantaneous position of the minor axis of the Earth's equatorial section.

The equations of motion of a geocentric satellite are determined as

$$\ddot{r} - r\dot{\vartheta}^2\cos^2\varphi - r\dot{\varphi}^2 + b\dot{r} = \frac{F_r}{M_s} = \frac{\partial \Delta}{\partial r}, \tag{1a}$$

$$\frac{1}{r\cos\varphi}\frac{d}{dt}(r^2\dot{\vartheta}\cos\varphi) + br\dot{\vartheta} = \frac{F_\vartheta}{M_s} = \frac{1}{r\cos\varphi}\frac{\partial \Delta}{\partial \vartheta}, \tag{1b}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi}) + r\dot{\vartheta}^2\cos\varphi\sin\varphi + br\dot{\varphi}sin\vartheta = \frac{F_\varphi}{M_s} = \frac{1}{r}\frac{\partial \Delta}{\partial \varphi}, \tag{1c}$$

where b is coefficient of resistive force per unit mass, M_s is mass of satellite, F_r, F_ϑ and F_ϕ are force components in the direction of r, n_1, and n_2, and Δ is the earth's gravitational potential, respectively, by using the procedure of Frick and Garber (1962)[7]

$$\Delta(r,\vartheta,\varphi) = \frac{g_0R_0^2}{r}\left[1 - \frac{J_2R_0^2}{r^2}\left(\frac{3\sin^2\varphi - 1}{2}\right)\right] + \frac{3g_0R_0^2}{r}\left(\frac{J_2^{(2)}}{r^2}\cos^2\varphi cos2\gamma\right), \tag{2}$$

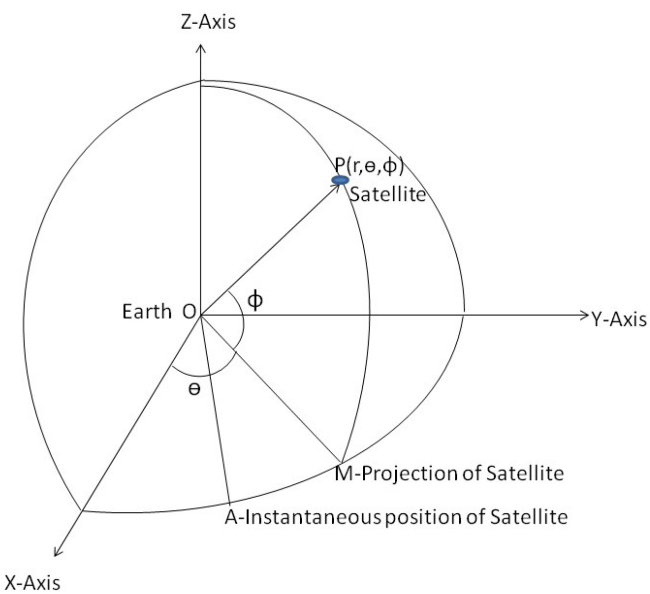


Figure 1: Configuration of a Geocentric Satellite

where g_0 is gravity on the surface of Earth, r is the redial distance from the center of the Earth to the satellite, J_2 is coefficient of oblateness of the Earth, R_0 is Earth's mean radius, $J_2^{(2)}$ is coefficient of Earth's EEE, ϕ is latitude of the satellite, ϑ is longitude of the satellite, and γ is earth's EEE parameter. The desired force components can be obtained from Eq.(2) as follows

$$\frac{F_r}{M_s} = \frac{\partial \Delta}{\partial r} = -\frac{g_0R_0^2}{r^2} + \frac{3}{2}\frac{J_2g_0R_0^4}{r^4}\left(3sin^2\varphi - 1\right) - 9\frac{J_2^{(2)}g_0R_0^4}{r^4}\cos^2\varphi\cos 2\gamma. \tag{3a}$$

$$\frac{F_\vartheta}{M_s} = \frac{1}{rcos\varphi}\frac{\partial \Delta}{\partial \vartheta} = -6\frac{g_0R_0^4}{r^4}J_2^2\cos\varphi\sin 2\gamma, \tag{3b}$$

$$\frac{F_\varphi}{M_s} = \frac{1}{r}\frac{\partial \Delta}{\partial \varphi} = -\frac{3}{2}\frac{J_2g_0R_0^4}{r^4}\sin 2\varphi - 3\frac{J_2^{(2)}g_0R_0^4}{r^4}\sin 2\varphi\cos 2\gamma. \tag{3c}$$

Substituting the expressions $\frac{\partial \Delta}{\partial r}$, $\frac{1}{rcos\varphi}\frac{\partial \Delta}{\partial \vartheta}$ and, $\frac{1}{r}\frac{\partial \Delta}{\partial \varphi}$ in Eqs.(1a), (1b) and (1c), we obtain

$$\ddot{r} - r\dot{\vartheta}^2\cos^2\varphi - r\dot{\varphi}^2 + b\dot{r} = -\frac{g_0R_0^2}{r^2} + \frac{3}{2}\frac{J_2g_0R_0^4}{r^4}(3sin^2\varphi - 1) - 9\frac{J_2^{(2)}g_0R_0^4}{r^4}\cos^2\varphi\cos 2\gamma, \tag{4a}$$

$$\frac{1}{r\cos\varphi}\frac{d}{dt}\left(r^2\dot{\vartheta}\cos\varphi\right) + br\dot{\vartheta} = -6\frac{g_0R_0^4}{r^4}J_2^2\cos\varphi\sin 2\gamma, \tag{4b}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\varphi}) + r\dot{\vartheta}^2\cos\varphi\sin\varphi + br\dot{\varphi}sin\vartheta = -\frac{3}{2}\frac{J_2g_0R_0^4}{r^4}\sin 2\varphi - 3\frac{J_2^{(2)}g_0R_0^4}{r^4}\sin 2\varphi\cos 2\gamma. \tag{4c}$$

There are three cases arise for solution procedure for the equations of motion of satellite under the gravitational attraction of the Earth:

(i) $b = 0$ and $\phi = 0$.

(ii) b 6= 0 and $\phi = 0$.

(iii) b = 06 and ϕ 6= 0.

Method of Analysis

Case I: Solution procedure when ϕ = 0 and b = 0: In this case, we assumed that satellite lies on equatorial plane and coefficient of resistive force is zero. By substituting $\phi = 0$ and $b = 0$ in Eqs.(4a), (4b) and 4c, we get

$$\ddot{r} - r\dot{\vartheta}^2 = -\frac{g_0R_0^2}{r^2} - 3\frac{J_2g_0R_0^4}{2r^4} - 9\frac{J_2^{(2)}g_0R_0^4}{r^4}\cos 2\gamma, \tag{5a}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\vartheta}\right) = -6\frac{J_2^{(2)}g_0R_0^4}{r^4}\sin 2\gamma. \tag{5b}$$

For the unperturbed system, $J_2 = 0$ and $J_2^{(2)} = 0$. From Eqs. (5a) and (5b), we deduce that

$$\ddot{r} - r\dot{\vartheta}^2 = -\frac{g_0R_0^2}{r^2}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\vartheta}\right) = 0. \qquad \text{(6a), (6b)}$$

Now, putting $u = \frac{1}{r}$ in Eqs.(6a) and (6b), we get

$$\frac{d^2u}{d\vartheta^2} + u = \frac{g_0R_0^2}{h^2}.$$

This is second order non-homogeneous equation with constant coefficients. Thus, the solution is given by

$$u(\vartheta) = \frac{g_0R_0^2}{h^2} + B\cos(\vartheta - \Omega),$$

where Ω and B are integration constants. Hence

$$r(\vartheta) = \frac{(h^2/g_0R_0^2)}{1 + (Ah^2/g_0R_0^2)\cos(\vartheta - \Omega)},$$

which is a polar equation of orbit and expressed as the equation of a conic section

$$r(\vartheta) = \frac{p}{1 + e\cos(\vartheta - \Omega)},$$

or

$$u(\vartheta) = \frac{1 + e\cos(\vartheta - \Omega)}{a(1 - e^2)}, \tag{7}$$

where $p = \frac{h^2}{g_0R_0^2} = a(1 - e^2)$ is semi-latus rectum, $e = \frac{Ah^2}{g_0R_0^2}$ is eccentricity, a is semi-major axis, and $r^2\dot{\vartheta} = h$ is angular momentum.

Now, substituting $r = \frac{1}{u}$, $\dot{r} = -h\frac{du}{d\vartheta}$, $\ddot{r} = -h^2u^2\frac{d^2u}{d\vartheta^2}$ and $\dot{\vartheta} = hu^2$ in Eq.(5a), we get

$$\frac{d^2u}{d\vartheta^2} + u = \frac{g_0R_0^2}{r^4\dot{\vartheta}^2} + \frac{3}{2}\frac{J_2g_0R_0^4u^2}{r^4\dot{\vartheta}^2} + 9\frac{J_2^{(2)}g_0R_0^4u^2}{r^4\dot{\vartheta}^2}\cos 2\gamma. \tag{8}$$

Replacing $\dot{\vartheta}$, r by their steady-state values $\dot{\vartheta}_0$ and r_0 in Eq.(8), we obtain

$$\begin{aligned}\frac{d^2u}{d\vartheta^2} + u =& \frac{g_0R_0^2}{r_0^4\dot{\vartheta}_0^2} + \frac{3}{2}\frac{J_2g_0R_0^4}{r_0^4\dot{\vartheta}_0^2a^2(1-e^2)^2}\left[1 + e\cos(\vartheta - \Omega)\right]^2 \\ &+ 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4\dot{\vartheta}_0^2a^2(1-e^2)^2}\left[1 + e\cos(\vartheta - \Omega)\right]^2\cos 2\gamma.\end{aligned} \tag{9}$$

We may take $\vartheta - \Omega = \dot{\vartheta}_0 t$, and $\gamma = \dot{\gamma} t$. From Eq.(9), we get

$$\begin{aligned}\frac{d^2u}{dt^2} + \dot{\vartheta}_0^2u =& \frac{g_0R_0^2}{r_0^4\dot{\vartheta}_0^2} + \frac{3}{2}\frac{J_2g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left[1 + e\cos\dot{\vartheta}_0t\right]^2 \\ &+ 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left[1 + e\cos\dot{\vartheta}_0t\right]^2\cos 2\dot{\gamma}t.\end{aligned} \tag{10}$$

The resonance in the satellite's motion occurs due to the frequencies $\dot{\vartheta}_0$ and $\dot{\gamma}$. We can find the resonance for neglecting the Earth's secular terms and oblateness J_2 in the equation (10). So, we obtain

$$\frac{d^2u}{dt^2} + \dot{\vartheta}_0^2u = 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left[1 + e\cos\dot{\vartheta}_0t\right]^2\cos 2\dot{\gamma}t. \tag{11}$$

Eq.(11) can be simplified as

$$\begin{aligned}\frac{d^2u}{dt^2} + \dot{\vartheta}_0^2u =& K_1\cos 2\dot{\gamma}t + K_2\left(\cos(2\dot{\vartheta}_0 + 2\dot{\gamma})t + \cos(2\dot{\vartheta}_0 - 2\dot{\gamma})t\right) \\ &+ K_3\left(\cos(\dot{\vartheta}_0 + 2\dot{\gamma})t + \cos(\dot{\vartheta}_0 - 2\dot{\gamma})t\right),\end{aligned} \tag{12}$$

where

$$K_1 = 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left(1 + \frac{e^2}{2}\right),\ K_2 = 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left(\frac{e^2}{4}\right),\ K_3 = 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}e.$$

A particular solution of Eq.(12) is

$$u_p(t) = A_1\cos 2\dot{\gamma}t + A_2\cos(2\dot{\vartheta}_0 - 2\dot{\gamma})t + A_3\cos(\dot{\vartheta}_0 - 2\dot{\gamma})t + A_4\cos(2\dot{\vartheta}_0 + 2\dot{\gamma})t + A_5\cos(\dot{\vartheta}_0 + 2\dot{\gamma})t, \tag{13}$$

where

$$A_1 = \frac{K_1}{-4\dot{\gamma}^2 + \dot{\vartheta}_0^2}$$

$$A_2 = \frac{K_2}{-(2\dot{\vartheta}_0 - 2\dot{\gamma})^2 + \dot{\vartheta}_0^2},$$

$$A_3 = \frac{K_3}{-(\dot{\vartheta}_0 - 2\dot{\gamma})^2 + \dot{\vartheta}_0^2},$$

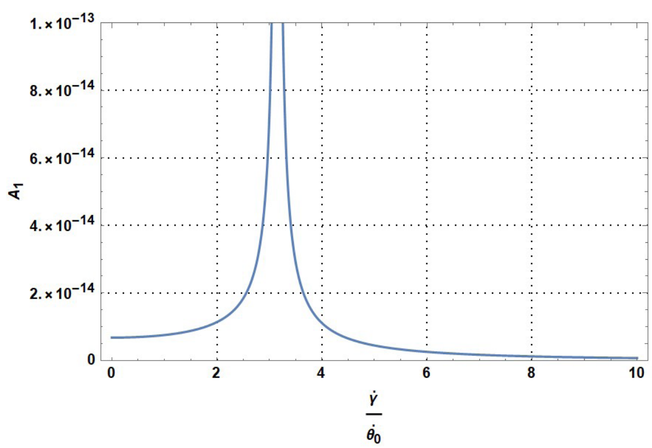
$$A_4 = \frac{K_2}{-(2\dot{\vartheta}_0 + 2\dot{\gamma})^2 + \dot{\vartheta}_0^2},$$

$$A_5 = \frac{K_3}{-(\dot{\vartheta}_0 + 2\dot{\gamma})^2 + \dot{\vartheta}_0^2},$$

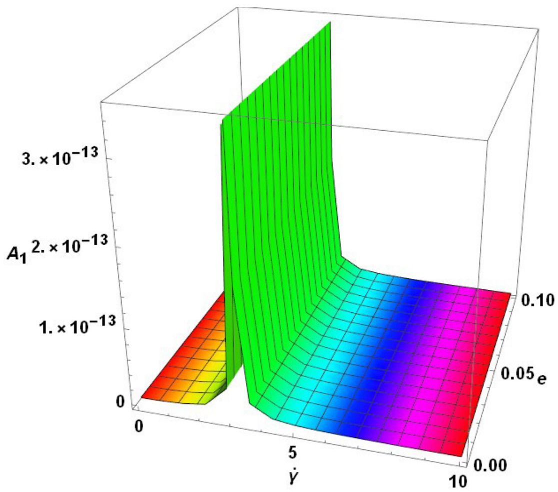
(14a) (14b) (14c) (14d) (14e)

Eqs.(14a)-(14e) are the oscillatory amplitudes.

Resonance: Resonance in a geocentric satellite's motion occurs due to the commensurablity between the frequencies $\dot{\gamma}$, $\dot{\vartheta}_0$. From the Eqs.(14a), (14b) and (14c), we examined that resonance appear in the satellite's motion at three resonant points $2\dot{\gamma} = \dot{\vartheta}_0$, $2\dot{\gamma} = 3\dot{\vartheta}_0$ and $\dot{\gamma} = \dot{\vartheta}_0$. Hence, the amplitudes $A_1 \to \infty$ for $2\dot{\gamma} \approx \dot{\vartheta}_0$, $A_2 \to \infty$ for $2\dot{\gamma} \approx 3\dot{\vartheta}_0$, $2\dot{\gamma} \approx \dot{\vartheta}_0$, and $A_3 \to \infty$ for $\dot{\gamma} \approx \dot{\vartheta}_0$.



(a)



(b)

Figure 2: Impact of orbital (e) and $\dot{\gamma}$ on oscillatory amplitude A_1 at a resonant point. $2\dot{\gamma} = \dot{\vartheta}_0$.

Case II: Solution procedure when $\boldsymbol{\phi} = \mathbf{0}$ and $\boldsymbol{b}$ **6= 0**

In this case, we assume that satellite lies on equatorial plane and the coefficient of resistive force is non-zero. By substituting $\phi = 0$ in Eq.(4a), we get

$$\ddot{r} - r\dot{\vartheta}^2 + b\dot{r} = -\frac{g_0 R_0^2}{r^2} - 3\frac{J_2 g_0 R_0^4}{2r^4} - 9\frac{J_2^{(2)} g_0 R_0^4}{r^4}\cos 2\gamma, \tag{15}$$

On substituting $r = \frac{1}{u}$, $\dot{r} = -h\frac{du}{d\vartheta}$, $\ddot{r} = -h^2u^2\frac{d^2u}{d\vartheta^2}$ and $\dot{\vartheta} = hu^2$ in Eq.(15), we get

$$\frac{d^2u}{d\vartheta^2} + \left(\frac{b}{\dot{\vartheta}}\right)\frac{du}{d\vartheta} + u = \frac{g_0R_0^2}{r^4\dot{\vartheta}^2} + \frac{3}{2}\frac{J_2g_0R_0^4u^2}{r^4\dot{\vartheta}^2} + 9\frac{J_2^{(2)}g_0R_0^4u^2}{r^4\dot{\vartheta}^2}\cos 2\gamma. \tag{16}$$

Replacing r, $\dot{\vartheta}$ by r_0, $\dot{\vartheta}_0$, and using the solution (7) in Eq. (16), we obtain

$$\frac{d^2u}{d\vartheta^2} + \left(\frac{b}{\dot{\vartheta}_0}\right)\frac{du}{d\vartheta} + u = \frac{g_0R_0^2}{r_0^4\dot{\vartheta}_0^2} + \frac{3}{2}\frac{J_2g_0R_0^4}{r_0^4\dot{\vartheta}_0^2a^2(1-e^2)^2}\left[(1+e\cos\dot{\vartheta}_0t)\right]^2 + 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4\dot{\vartheta}_0^2a^2(1-e^2)^2}\left[1+e\cos\dot{\vartheta}_0t\right]^2\cos 2\gamma. \tag{17}$$

We assume that $\vartheta - \Omega = \dot{\vartheta}_0 t$ and $\gamma = \gamma t$, and using these values in Eq.(17), we get

$$\frac{d^2u}{dt^2} + (b)\frac{du}{dt} + \dot{\vartheta}_0{}^2u = \frac{g_0R_0^2}{r_0^4} + \frac{3}{2}\frac{J_2g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left[1+e\cos\dot{\vartheta}_0t\right]^2 + 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left[1+e\cos\dot{\vartheta}_0t\right]^2\cos 2\gamma. \tag{18}$$

The resonance appears between the rate of change of EEE and angular velocity of the satellite, including the coefficient of resistive force. For examining the resonance, we are ignoring the secular and oblateness J_2 terms in the equation (18). So, we can write

$$\frac{d^2u}{dt^2} + (b)\frac{du}{dt} + \dot{\vartheta}_0{}^2u = 9\frac{J_2^{(2)}g_0R_0^4}{r_0^4a^2(1-e^2)^2}\left[1+e\cos\dot{\vartheta}_0t\right]^2\cos 2\gamma. \tag{19}$$

Eq.(19) can be simplified as

$$\frac{d^2u}{dt^2} + (b)\frac{du}{dt} + \dot{\vartheta}_0{}^2u = K_1\cos 2\dot{\gamma}t + K_2\left(\cos(2\dot{\vartheta}_0+2\dot{\gamma})t + \cos(2\dot{\vartheta}_0-2\dot{\gamma})t\right) + K_3\left(\cos(\dot{\vartheta}_0+2\dot{\gamma})t + \cos(\dot{\vartheta}_0-2\dot{\gamma})t\right), \tag{20}$$

A particular solution of Eq.(20) is $u_{(b)p}(t) = A_1\cos(2\dot{\gamma}t - \alpha_1) + A_2\cos((2\dot{\vartheta}_0 - 2\dot{\gamma}t) - \alpha_2) + A_3\cos((\dot{\vartheta}_0 - 2\dot{\gamma}t) - \alpha_3)$

$$+ A_4\cos((2\dot{\vartheta}_0 + 2\dot{\gamma}t) - \alpha_4) + A_5\cos((\dot{\vartheta}_0 + 2\dot{\gamma}t) - \alpha_5), \tag{21}$$

where

$$\alpha_{(b)1} = \tan^{-1}\left[\frac{2b\dot{\gamma}}{-4\dot{\gamma}^2+\dot{\vartheta}_0{}^2}\right], \alpha_{(b)2} = \tan^{-1}\left[\frac{b(2\dot{\vartheta}_0-2\dot{\gamma})}{\dot{\vartheta}_0^2-(2\dot{\vartheta}_0-2\dot{\gamma})^2}\right], \alpha_{(b)3} = \tan^{-1}\left[\frac{b(\dot{\vartheta}_0-2\dot{\gamma})}{\dot{\vartheta}_0^2-(\dot{\vartheta}_0-2\dot{\gamma})^2}\right]$$

$$\alpha_{(b)4} = \tan^{-1}\left[\frac{b(2\dot{\vartheta}_0+2\dot{\gamma})}{\dot{\vartheta}_0^2-(2\dot{\vartheta}_0+2\dot{\gamma})^2}\right], \alpha_{(b)5} = \tan^{-1}\left[\frac{b(\dot{\vartheta}_0+2\dot{\gamma})}{\dot{\vartheta}_0^2-(\dot{\vartheta}_0+2\dot{\gamma})^2}\right]. ,$$

are phase angles, and

$$A_{(b)1} = \frac{K_1}{\sqrt{4b^2\dot{\gamma}^2 + (\dot{\vartheta}_0{}^2 - 4\dot{\gamma}^2)^2}}$$

$$A_{(b)2} = \frac{K_2}{\sqrt{b^2(2\dot{\vartheta}_0 - 2\dot{\gamma})^2 + [\dot{\vartheta}_0{}^2 - (2\dot{\vartheta}_0 - 2\dot{\gamma})^2]^2}},$$

$$A_{(b)3} = \frac{K_3}{\sqrt{b^2(\dot{\vartheta}_0 - 2\dot{\gamma})^2 + [\dot{\vartheta}_0{}^2 + (\dot{\vartheta}_0 - 2\dot{\gamma})^2]^2}},$$

$$A_{(b)4} = \frac{K_2}{\sqrt{b^2(2\dot{\vartheta}_0 + 2\dot{\gamma})^2 + [\dot{\vartheta}_0{}^2 - (2\dot{\vartheta}_0 + 2\dot{\gamma})^2]^2}},$$

$$A_{(b)5} = \frac{K_3}{\sqrt{b^2(\dot{\vartheta}_0 + 2\dot{\gamma})^2 + [\dot{\vartheta}_0{}^2 + (\dot{\vartheta}_0 + 2\dot{\gamma})^2]^2}},$$

(22a) (22b) (22c) (22d) (22e)

Eqs.(22a)-(22e) are oscillatory amplitudes with coefficient of resistive force.

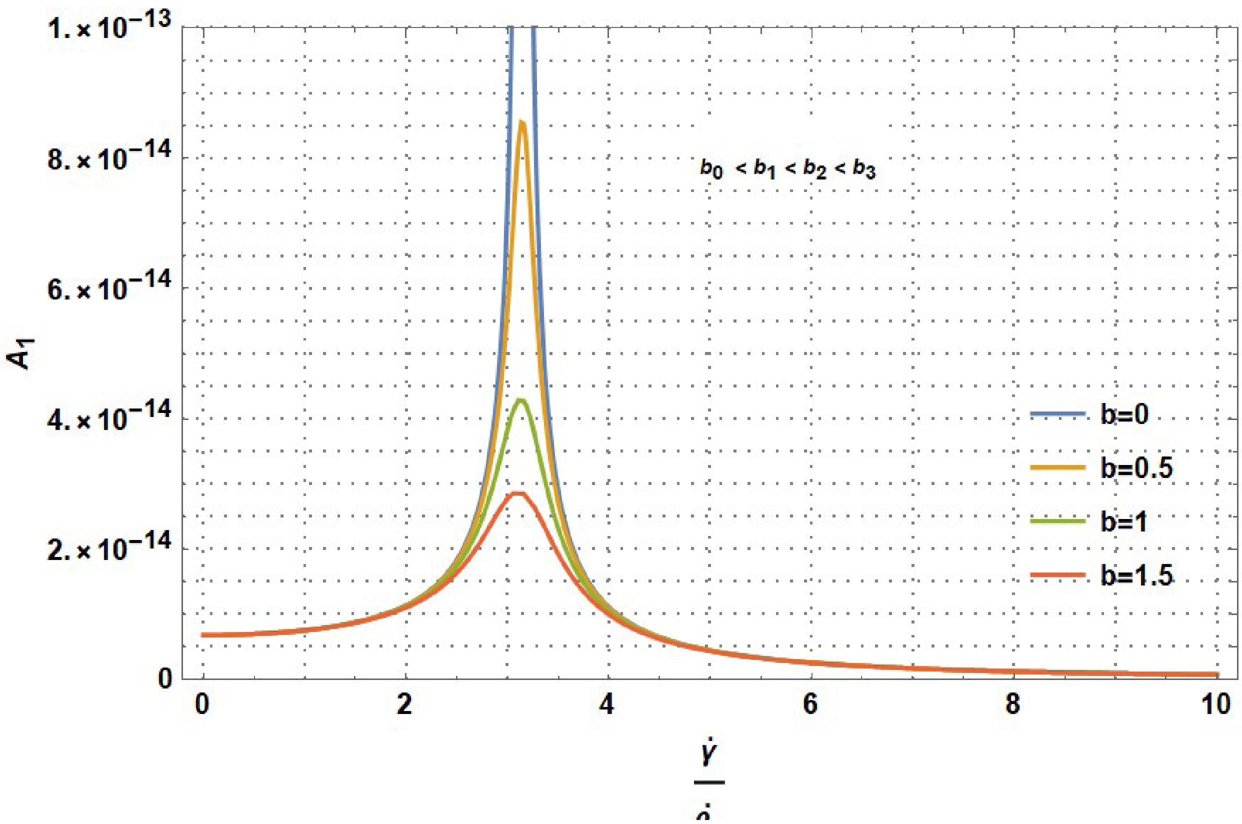
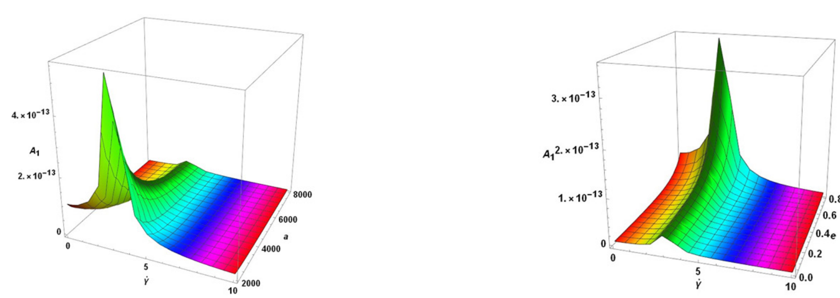


Figure 3: Effect of b and $\dot{\gamma}$ on resonance curve for the oscillatory amplitude $A_{(b)1}$.

General case: (when ϕ 6= 0 and b 6= 0): In this case, we assume that the satellite does not lie on equatorial plane and the coefficient of resistive force is non-zero. Substituting $\dot{\phi} = 0$ in Eqs. (4a) (assuming ϕ is independent of time t and constant), we get

(a) Effect of a and $\dot{\gamma}$ on $A_{(b)1}$. (b) Effect of e and $\dot{\gamma}$ on $A_{(b)1}$.



(a) Effect of a and $\dot{\gamma}$ on $A_{(b)1}$. (b) Effect of e and $\dot{\gamma}$ on $A_{(b)1}$.

Figure 4: Impact of eccentricity (e), coefficient of resistive force (b) and $\dot{\gamma}$ on oscillatory amplitude $A(b)$1.

$$\ddot{r} - r\dot{\vartheta}^2\cos^2\varphi + b\dot{r} = -\frac{g_0R_0^2}{r^2} + \frac{3}{2}\frac{J_2g_0R_0^4}{r^4}\left(3sin^2\varphi - 1\right) - 9\frac{J_2^{(2)}g_0R_0^4}{r^4}\cos^2\varphi\cos 2\gamma, \tag{23}$$

On substituting $u = \frac{1}{r}$, $\dot{r} = -\left(\frac{h}{\cos\varphi}\right)\frac{du}{d\vartheta}$ and $r = -\frac{h^2u^2}{\cos^2\varphi}\frac{d^2u}{d\vartheta^2}$ in Eq.(23), we get

$$\frac{d^2u}{d\vartheta^2} + \left(\frac{b}{\dot{\vartheta}\cos\varphi}\right)\frac{du}{d\vartheta} + \cos^2\varphi u = \frac{g_0R_0^2}{r^4\dot{\vartheta}^2} + \frac{3}{2}\frac{J_2g_0R_0^4u^2}{r^4\dot{\vartheta}^2} + 9\frac{J_2^{(2)}g_0R_0^4u^2}{r^4\dot{\vartheta}^2}\cos 2\gamma. \tag{24}$$

Replacing r, $\vartheta^{\cdot}$ by their steady-state values r_0 and $\vartheta^{\cdot}_0$ in Eq.(24), we obtain

$$\frac{d^2u}{d\vartheta^2} + \left(\frac{b}{\dot{\vartheta}_0 \cos\varphi}\right)\frac{du}{d\vartheta} + \cos^2\varphi u = \frac{g_0 R_0^2}{r_0^4 \dot{\vartheta}_0^2} + \frac{3}{2}\frac{J_2 g_0 R_0^4}{r_0^4 \dot{\vartheta}_0^2 a^2 (1-e^2)^2}\left[(1 + e\cos\dot{\vartheta}_0 t)\right]^2 + 9\frac{J_2^{(2)} g_0 R_0^4}{r_0^4 \dot{\vartheta}_0^2 a^2(1-e^2)^2}\left[1 + e\cos\dot{\vartheta}_0 t\right]^2 \cos 2\gamma. \tag{25}$$

We may take $\vartheta = \vartheta^{\cdot}_0 t$ and $\gamma = \gamma t,^{\cdot}$ in Eq. (25). Since we are examining the resonance in the satellite's motion between the rate of change of EEE and angular velocity of the satellite with the coefficient of resistive force and latitude of the satellite. To analyze the resonant points, ignoring the secular and oblateness J_2 in the equation (25); we get

$$\frac{d^2u}{dt^2} + \left(\frac{b}{cos^2\varphi}\right)\frac{du}{dt} + \dot{\vartheta_0}^2 u = 9\frac{J_2^{(2)} g_0 R_0^4}{r_0^4 a^2 (1-e^2)^2}\cos^2\varphi\left[1 + e\cos\dot{\vartheta}_0 t\right]^2 \cos 2\gamma. \tag{26}$$

Eq.(26) can be simplified as

$$\frac{d^2u}{dt^2} + \left(\frac{b}{cos^2\varphi}\right)\frac{du}{dt} + \dot{\vartheta_0}^2 u = l_1 \cos 2\dot{\gamma}t + l_2\left(\cos(2\dot{\vartheta}_0 + 2\dot{\gamma})t + \cos(2\dot{\vartheta}_0 - 2\dot{\gamma})t\right) + l_3\left(\cos(\dot{\vartheta}_0 + 2\dot{\gamma})t + \cos(\dot{\vartheta}_0 - 2\dot{\gamma})t\right), \tag{27}$$

where $l_1 = 9\frac{J_2^{(2)} g_0 R_0^4}{r_0^4 a^2(1-e^2)^2}\left(1 + \frac{e^2}{2}\right)\cos^2\varphi$, $l_2 = 9\frac{J_2^{(2)} g_0 R_0^4}{r_0^4 a^2(1-e^2)^2}\left(\frac{e^2}{4}\right)\cos^2\varphi$, $l_3 = 9\frac{J_2^{(2)} g_0 R_0^4}{r_0^4 a^2(1-e^2)^2} e\cos^2\varphi$.

A particular solution of Eq. (27) is $u(\phi)p(t) = A(\phi)1\ \cos(2^{\cdot}\gamma t - \alpha(\phi)1) + A(\phi)2\ \cos((2\vartheta^{\cdot}0 - 2\gamma t^{\cdot}) - \alpha(\phi)2) + A(\phi)3\ \cos((\vartheta^{\cdot}0 - 2\gamma t^{\cdot}) - \alpha(\phi)3) + A(\phi)4\ \cos((\vartheta^{\cdot}0 + 2\gamma t^{\cdot}) - \alpha(\phi)4) + A(\phi)5\ \cos((2\vartheta^{\cdot}0 + 2\gamma t^{\cdot}) - \alpha(\phi)5)$ (28)

where

$$\alpha_{(\varphi)1} = \tan^{-1}\left[\frac{2(\frac{b}{cos\varphi})\dot{\gamma}}{-4\dot{\gamma}^2 + \dot{\vartheta_0}^2}\right], \alpha_{(\varphi)2} = \tan^{-1}\left[\frac{(\frac{b}{cos\varphi})(2\dot{\vartheta}_0 - 2\dot{\gamma})}{\dot{\vartheta}_0^2 - (2\dot{\vartheta}_0 - 2\dot{\gamma})^2}\right], \alpha_{(\varphi)3} = \tan^{-1}\left[\frac{(\frac{b}{cos\varphi})(\dot{\vartheta}_0 - 2\dot{\gamma})}{\dot{\vartheta}_0^2 - (\dot{\vartheta}_0 - 2\dot{\gamma})^2}\right]$$

$$\alpha_{(\varphi)4} = \tan^{-1}\left[\frac{(\frac{b}{cos\varphi})(\dot{\vartheta}_0 + 2\dot{\gamma})}{\dot{\vartheta}_0^2 - (\dot{\vartheta}_0 + 2\dot{\gamma})^2}\right], \alpha_{(\varphi)5} = \tan^{-1}\left[\frac{(\frac{b}{cos\varphi})(2\dot{\vartheta}_0 + 2\dot{\gamma})}{\dot{\vartheta}_0^2 - (2\dot{\vartheta}_0 + 2\dot{\gamma})^2}\right],$$

are phase angles, and

$$A_{(\varphi)1} = \frac{l_1}{\sqrt{4(\frac{b}{\cos\varphi})^2\dot{\gamma}^2 + (\dot{\vartheta_0}^2 - 4\dot{\gamma}^2)^2}}$$

$$A_{(\varphi)2} = \frac{l_2}{\sqrt{(\frac{b}{\cos\varphi})^2(2\dot{\vartheta}_0 - 2\dot{\gamma})^2 + [\dot{\vartheta_0}^2 - (2\dot{\vartheta}_0 - 2\dot{\gamma})^2]^2}},$$

$$A_{(\varphi)3} = \frac{l_3}{\sqrt{(\frac{b}{\cos\varphi})^2(\dot{\vartheta}_0 - 2\dot{\gamma})^2 + [\dot{\vartheta_0}^2 + (\dot{\vartheta}_0 - 2\dot{\gamma})^2]^2}},$$

$$A_{(\varphi)4} = \frac{l_3}{\sqrt{(\frac{b}{\cos\varphi})^2(\dot{\vartheta}_0 + 2\dot{\gamma})^2 + [\dot{\vartheta_0}^2 + (\dot{\vartheta}_0 + 2\dot{\gamma})^2]^2}},$$

$$A_{(\varphi)5} = \frac{l_2}{\sqrt{(\frac{b}{\cos\varphi})^2(2\dot{\vartheta}_0 + 2\dot{\gamma})^2 + [\dot{\vartheta_0}^2 - (2\dot{\vartheta}_0 + 2\dot{\gamma})^2]^2}},$$

(29a) (29b) (29c) (29d) (29e)

Eqs. (29a)-(29e) are oscillatory amplitudes with latitude of satellite and resistive force coefficient.

Conclusion

In this study, we have expressed the equations of motion of the satellite in a spherical coordinate system by using the Earth's potential. The unperturbed solution of the satellite's motion has been used to reduce the system of equations into a second-order ODE. We obtained the solution for the established second-order ODE with the three cases: i) $b = 0$ and $\phi = 0$, ii) b 6= 0 and $\phi = 0$, and iii) b 6= 0 and ϕ 6= 0. The resonances in the satellite's motion have been obtained concerning the frequencies. We have shown the effect of EEEP in the motion of the satellite, eccentricity (e), semi-major axis (a), and the resistive force on the oscillatory amplitudes through the dynamics in 2D and 3D as per the suitability.

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