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A Bernstein Polynomial Differential Quadrature Method for Numerical Solutions of Kawahara Equation

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Abstract

A Bernstein polynomial differential quadrature method (BDQM) has been applied to study Kawahara equation numerically. Bernstein polynomials have been used as base functions to find weighting coefficients. After discretization via differential quadrature method, a system of ordinary differential equations is obtained which has been solved by Runge-Kutta method. This is a simple and straightforward method which gives very good results even for higher order partial differential equations.

Keywords: *Kawahara Equation, Differential Quadrature method, Bernstein Polynomial, Runge Kutta method.*

Introduction

Consider a fifth order partial differential equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}-\frac{\partial^5 u}{\partial x^5}=0 \qquad (1.1)$$

with the Dirichlet boundary conditions. Initial condition is given by

$$u(x, 0) = f(x) \qquad (1.2)$$

Above equation is known as Kawahara equation. Kawahara equation occurs in plasmas[1] to model shallow water waves and magneto acoustic waves. Kawahara equation shows complex dynamical behaviour. This equation was introduced by Kawahara[2] to study shallow water waves. It occurs in flame propagation dynamics[3,4] two-phase flows in cylindrical or plane geometries, surface film-flows.[5,6,7]

This equation is non integrable. There are some analytical solutions which have been obtained for special cases. Nagashima[8] studied solitary waves formed by this equation at different time levels. Kashkari applied Laplace homotopy perturbation method for numerical study of Kawahara equation.[9] Haq et.al presented meshless method of lines for numerical solutions of Kawahara type equations.[10] Suarez and Morales[11] applied Fourier splitting method to solve Kawahara equation. Rashidinia and Rasaulizadeh applied a local RBF method to solve Kawahara equation. There are some more numerical methods which have been applied by some researchers to solve Kawahara equation[12].[13] Differential quadrature method has a wide area of research. B-spline differential quadrature,[14] sinc differential quadrature,[15] cosine differential quadrature,[16] polynomial dif- ferential quadrature,[17] trigonometric B-spline differential quadrature[18] and exponential B- spline differential quadrature method[19] have been studied by different researchers to study partial differential equations in one, two and three dimensional space.

Saif and Saadawi[20] used Bernstein polynomial based differential quadrature method (BDQM) to study convection diffusion equation. They derived differential quadrature scheme following the idea of Lagrange polynomial differential quadrature method to find weighting coefficients. An ADI differential quadrature method is also applied by Saif and Saadawi[2122] to solve unsteady flow of polytropic gas and two dimensional convection diffusion equations. We have developed a more

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straight forward approach to find weighting coefficients. Bernstein polynomial differential quadrature method has been used to study one dimensional Burgers' and Fisher's equation.[23] In this work, we have extended our previous work by applying BDQM to a fifth order Kawahara equation and we observed that method is performing very well even for higher dimensional equations.

The structure of the paper has been organized as follows. Section 2 contains description of Bernstein polynomials followed by section 3 which describes Bernstein differential quadrature method and evaluation of weighting coefficients. Stability has been discussed in section 4. Section 5 contains some numerical results to demonstrate and validate ef ficiency of the present method. Section 6 contains remarks drawn from the computed results and from the performance of the Bernstein differential quadrature method.

The Bernstein Polynomial

The Bernstein polynomial is a linear combination of its basis polynomials. The name of Bernstein polynomials is originated from the name of Sergei Natanowich Bernstein. A degree n Bernstein basis polynomial on [0, 1] is defined by Singh[25] as

$$B_{k,n}(x) = \binom{n}{k} x^k(1-x)^{n-k}, \quad 0 \le k \le n \tag{2.1}$$

coefficients of binomial expansion are defined as usual. Obviously, n-th degree Bernstein polynomials are n+1 in number. We take $B_{k,n}(x) = 0$ if $k < 0$ or $k > n$, for mathematical convenience.

Some properties of Bernstein Polynomial. The Bernstein polynomial has following properties[2627]

(a) Nonnegative: $B_{k,n}(x) \geq 0$ for x [0, 1].

(b) Unity Partision Property: We can all Bernstein basis polynomials to get 1.

$$\sum_{k=0}^{n} B_{k,n}(x) = 1. \tag{2.2}$$

(c) Recursion relation formula:

$$B_{k,n}(x) = [xB_{k-1,n-1}(x) + (1-x)B_{k,n-1}(x)] \tag{2.3}$$

(d) Degree raising property: An $(n-1)$ degree polynomial can be written as a linear combination of polynomials of degree n as follows

$$B_{k,n-1}(x) = \frac{k+1}{n} B_{k+1,n}(x) + \frac{n-k}{n} B_{k,n}(x). \tag{2.4}$$

(e) We have

$$B_n(f)(x) = \sum_{k=0}^{n} f(k/n)B_{k,n}(x) \tag{2.5}$$

converges to $f(x)$ uniformly on [0,1] as $n \to \infty$, where f(x) $\in c[0, 1]$

(f) Derivative: We can express first derivative of Bernstein polynomial as follows

$$B'_{k,n}(x) = n[B_{k-1,n-1}(x) - B_{k,n-1}(x)] \tag{2.6}$$

Above formula has been modified as given in[28] as follows,

$$B'_{k,n}(x) = (n-k+1)B_{k-1,n}(x) + (2k-n)B_{k,n}(x) - (k+1)B_{k+1,n}(x). \tag{2.7}$$

Bernstein Differential Quadrature Method (BDQM)

Differential quadrature approximation for the p-th partial derivative of an unknown function $u(x, t)$ can be represented by the formula[29]

$$u_x^{p}(x_i, t) = \sum_{j=0}^{n} w_{ij}^{(p)} u(x_j, t), \quad i = 0, 1, 2 \ldots n, \tag{3.1}$$

here $w^{(p)}$ are p-th order weighting coefficients and $u^p(x_i, t)$ stands for p-th order derivative of $u(x_i, t)$ with respect to x at x_i. An algorithm for evaluation of weighting coefficients was first of all presented by Bellman and his fellow researchers.[29] Quan and Chang[30],[31] and Shu and Richards[32] proposed a recursion formula to find weighting coefficients. This formula is independent of the number and positioning of nodes. In BDQM, Bernstein polynomials have been applied for space discretization. This gives a system ODE'S. SSPRK-43 method is applied to get the final solution of system of ODE's. We find first order weighting coefficients by applying Bernstein polynomials as test function. In BDQM, first order coefficients are calculated as

$$w_{ij}^{(1)} = (n-j+1)B_{j-1,n}(x_i) + (2j-n)B_{j,n}(x_i) - (j+1)B_{j+1,n}(x_i). \tag{3.2}$$

Higher order weighting coefficients are calculated by using Shu's formula[32] given by

$$w_{ij}^{(p)} = p\left[w_{ij}^{(1)} w_{ii}^{(p-1)} - \frac{w_{ij}^{(p-1)}}{(x_i - x_j)}\right] \quad for\ i \neq j, \tag{3.3}$$

$$w_{ii}^{(p)} = -\sum_{j=0,\, j\neq i}^{n} w_{ij}^{(p)} \qquad for\ i = j, \tag{3.4}$$

from this formula we can evaluate weighting coefficients of higher orders very easily. We can also use Bernstein polynomial to evaluate higher order weighting coefficients.

Since Shu formula is more straightforward and reduces the size of computational efforts, weighting coefficients upto fifth order have been determined by above recurrence formula.

Discretizing (1.1) by Bernstein differential quadrature method and taking boundary conditions into consideration, we get

$$\frac{du_i}{dt} = -u_i \sum_{j=1}^{n-1} w_{ij}^{(1)} u(x,t) - \sum_{j=1}^{n-1} w_{ij}^{(3)} u(x,t) + \sum_{j=1}^{n-1} w_{ij}^{(5)} u(x,t) + F, \quad i = 1, 2 \ldots n-1 \quad (3.5)$$

where F is the part containing boundary conditions of the problem. Above is a system of ordinary differential equations which can be solved by Runge Kutta method to get the final solution at the knots.

Stability Analysis

Consider the problem

$$\frac{\partial u}{\partial t} = g(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{xxxxx}) \quad (4.1)$$

By semi-discretization in space variable *x* by differential quadrature method, the following system of ordinary differential equations is obtained

$$\frac{d[u]}{dt} = [A][u] + [c] \quad (4.2)$$

where [*u*] is unknown vector of functional values at grid points, [*c*] contains nonhomo- geneous part and boundary conditions, and A is resultant coefficient matrix. In order to discuss the stability of derived scheme for (1.1), we linearize it by assuming $u(xi, t) = \kappa$ in the nonlinear terms.

Now the coefficient matrix [*A*] becomes as follows

$$A_{ij} = -\kappa w_{ij}^{(1)} - w_{ij}^{(3)} + w_{ij}^{(5)}$$

The stability of this system depends on eigenvalues of A. As t , for the stable solution

of u we must have[33]

(a) $-2.78 < \Delta t\lambda i < 0$;, if all eigenvalues are real

(b) $-2\sqrt{2} < \Delta t\lambda i < 2\sqrt{2}$, if eigenvalues have only complex components

(c) $\Delta t\lambda i$ should be in a region as shown by the Figure 1 if eigenvalues are complex.

where λi's are eigenvalues of A and Δt is the time step. Distribution of $\Delta t\lambda i$ is depicted in Figure 2. It may be noticed that eigenvalues lie within the stability region.

Numerical Experiments and Results

The numerical method based on Bernstein polynomials has been applied on two impor- tant problems to validate the applicability of the proposed method. Accuracy has been checked by finding maximum absolute norms as follows

$$L_\infty = |\,|u^{exact} - u^N|\,|_\infty = max|u_i^{exact} - u_i^N| \quad (5.1)$$

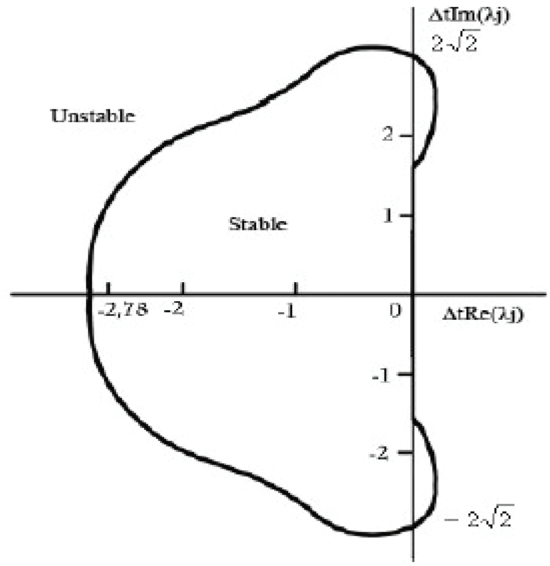

Figure 1. Stability region when eigenvalues are complex

here u^N stands for numerical solution. u^{exact} and u^N represent analytical and approximate solutions respectively at the knot *xi*. The values of the parameters c and k are defined by

$$c = \frac{36}{169} \text{ and } k = \frac{1}{2\sqrt{13}} \quad (5.2)$$

Example 1: Consider (1.1) with the following exact solution[24]

$$u(x,t) = \frac{-72}{169} + \frac{105}{169} \sec h^4[k(x+ct)],\ 0 < x < 60,\ t > 0 \quad (5.3)$$

Initial condition is defined by

$$u(x,0) = \frac{-72}{169} + \frac{105}{169} \sec h^4(kx) \quad (5.4)$$

and boundary conditions are

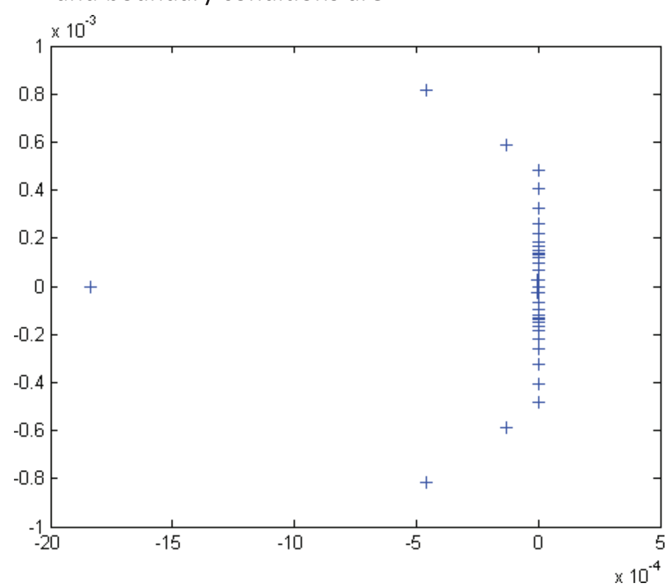

Figure 2. Plot of $\Delta t\lambda i$ for $\Delta t = 0.01$

$$u(0,t) = \frac{-72}{169} + \frac{105}{169}\sec h^4(kct) \qquad (5.5)$$

and

$$u(60,t) = \frac{-72}{169} + \frac{105}{169}\sec h^4[k(60+ct)] \qquad (5.6)$$

Computed numerical solutions for the present problem have been presented in Table 1. Method has been compared with the multi quadric radial basis functions method. It may be noticed that Bernstein differential quadrature method is giving better results. In Table 2, solutions have been given at different time levels. Numerical and exact solutions for t 1 have been depicted in Figures 3 and 4. It may be seen that approximate solution are very close to exact solutions.

Example 2: Consider Kawahara equation with the following exact solution[24]

$$u(x,t) = \frac{-72}{169} + \frac{420}{169}\frac{\sec h^2[k(x+ct)]}{1+\sec h^2[k(x+ct)]}, \; -30 < x < 60,\; t > 0 \qquad (5.7)$$

Boundary conditions have been taken from the exact solution and initial condition is

$$u(x,0) = \frac{-72}{169} + \frac{420}{169}\frac{\sec h^2(kx)}{1+\sec h^2(kx)}, \; -30 < x < 60,\; t > 0 \qquad (5.8)$$

Table 1

x	Present method	RBF[24]
0	1.09E-05	2.42E-01
10	6.60E-03	2.17E-02
20	2.53E-04	8.91E-02
30	2.54E-06	5.38E-03
40	1.91E-08	3.55E-04
50	1.16E-10	3.07E-05
60	5.55E-17	4.98E-05

Numerical solution of Example 1 at t = 0.5 with Δt = 0.01 and n = 60

Table 2

x	t = 0.1	t = 0.2	t = 1.0
0	2.27E-06	6.46E-012	2.75E-05
10	4.49E-03	7.30E-03	1.94E-02
20	1.18E-04	2.22E-04	1.49E-03
30	7.95E-07	1.31E-06	3.20E-05
40	4.02E-09	1.07E-08	5.00E-07
50	1.79E-11	5.28E-11	6.85E-09
60	5.55E-17	1.00E-17	5.55E-17

Maximum absolute errors of Example 1 with Δt = 0.01 and n = 90

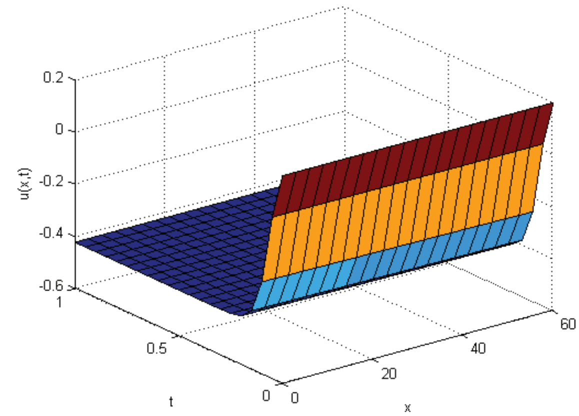

Figure 3. Plot of nu- merical solutions of Ex- ample 1 for t ≤ 1

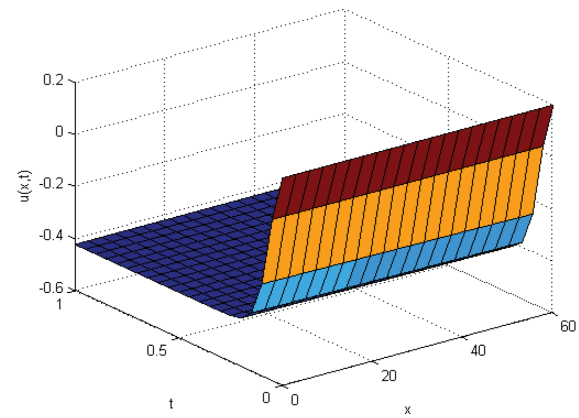

Figure 4. Plot of exact solutions of Example 1 for t ≤ 1

We have computed solutions for t = 1 for this problem. Maximum absolute errors have been compared with those obtained by.[24] It may be noticed from Table 3 that our solutions are better than the solution of RBF method. Maximum absolute errors have been given in Table 4 at different time levels for the domain $-30 < x < 60$. Numerical and exact solutions have been depicted in Figures 5 and 6 at time t = 0.5 which show a traveling wave moving in the x direction.

Table 3.

x	Present method	RBF[24]
0	6.06E-03	3.79E-01
10	2.92E-03	8.40E-03
20	2.23E-03	1.70E-02
30	3.42E-04	5.43E-04
40	3.48E-05	3.08E-05
50	3.62E-06	1.96E-04

Numerical solution of Example 2 at t = 1.0 with Δt = 0.01 and n = 90

Table 4

x	t = 0.1	t = 0.2	t = 0.3	t = 0.4	t = 0.5
–30	1.43E-06	2.15E-12	1.45E-06	1.46E-06	1.47E-06
–20	1.10E-03	2.06E-03	3.21E-06	4.29E-03	5.40E-03
–10	6.45E-05	2.28E-04	1.54E-04	2.62E-04	3.74E-04
0.0	3.76E-04	7.39E-04	1.24E-03	1.76E-03	2.33E-03
10	1.71E-04	6.55E-04	8.13E-04	1.12E-03	1.43E-03
20	2.31E-04	4.58E-04	6.86E-04	9.11E-04	1.13E-04
30	3.86E-05	7.23E-05	1.10E-04	1.45E-04	1.80E-04
40	3.73E-06	6.94E-06	1.07E-05	1.41E-05	1.76E-05
50	3.76E-07	6.96E-07	1.07E-06	1.43E-06	1.79E-06
60	3.46E-10	5.00E-10	3.42E-10	3.40E-10	3.38E-10

Numerical solution of Example 2 at different time levels with $\Delta t = 0.01$ and $n = 90$

Conclusion

Present method demonstrates application of Bernstein polynomial differential quadrature method to solve fifth order partial differential equation. Obtained results are satisfactory and better than the results found in literature. Bernstein basis polynomials have been used to find weighting coefficients. Differential quadrature method has been applied to discretize the given equation and to obtain a system of ordinary differential equation. Resulting system has been solved by Runge Kutta method. This method is very simple to apply and gives solution with less computational efforts. Method can be modified to solve problems arising in physics and engineering areas.

Conflicts of interest statement - The authors declare that they have no conflict of interest.

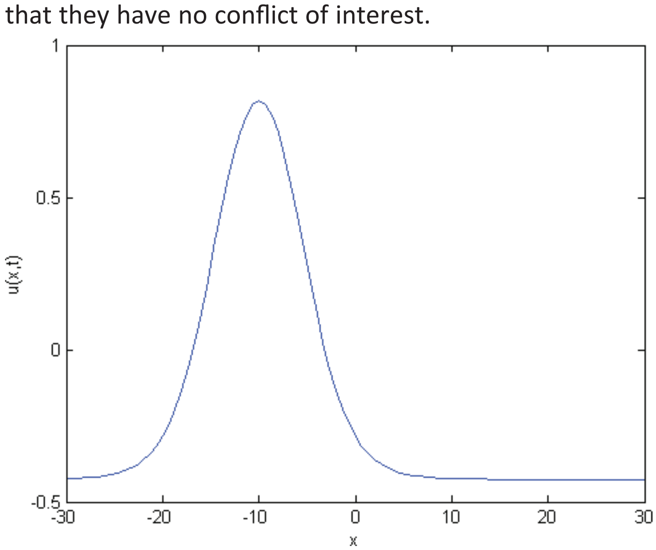

Figure 5. Plot of nu- merical solutions of Ex- ample 2 for t = 0.5

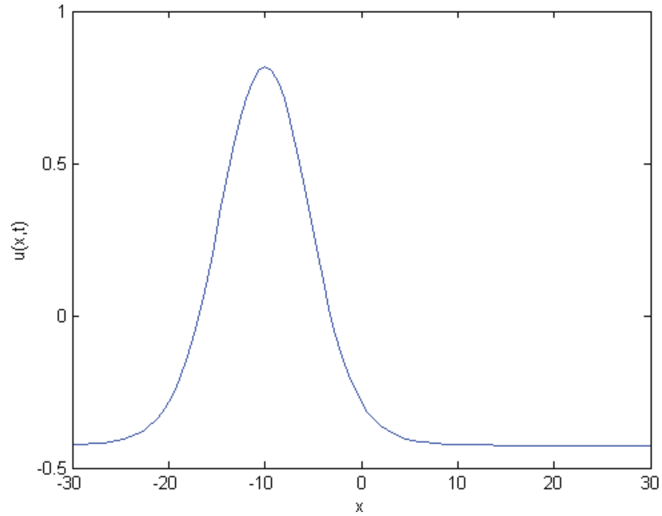

Figure 6. Plot of exact solutions of Example 2 for t = 0.5

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