ORIGINAL ARTICLE

INTERNATIONAL JOURNAL OF CONVERGENCE IN HEALTHCARE

IJCIH
Converging Healthcare & Technology

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IJCIH & Pratyaksh Medicare LLP

www.ijcih.com
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Numerical Solution of Korteweg-De Vries Equation by Quintic B-Spline Differential Quadrature Method

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Abstract

In this manuscript, we have used quintic B-spline functions to get the numerical solutions of the Korteweg-De Vries (KdV) equation, which is a nonlinear equation. The functions and their derivatives have been approximated using the quintic B-spline function. By using the Thomas algorithm, we obtained the weighting coefficients of the differential quadrature method, and the SSP-RK43 scheme has been used to solve differential equations. The test problem has been solved numerically to demonstrate the effectiveness and accuracy of the method. Numerical solutions have been presented in tables and illustrated graphically.

Keywords: *Korteweg-De Vries equation, Quintic B-spline function, Shu's recurrence, Thomas algorithm, SSP-RK43 scheme*

Introduction

Consider the Korteweg-de Vries (KdV) equation

$$V_t + \varepsilon V V_z + \mu V_{zzz} = 0 \tag{1.1}$$

here and are real constants. The KdV equation was introduced by Korteweg and de Vries in 1895[1], which is a nonlinear partial differential equation. This equation was used to describe many physical phenomena such as shallow water waves[2], beam propagation[3], bubble-liquid mixtures[4], ion-acoustic waves[5], fluid mechanics[6], and other areas.

The KdV equation has become a very popular research topic due to its many applications and occurrences in the real world. To obtain the solutions to the KdV equation, many researchers have used various techniques. The numerical solutions of the KdV equation were discussed by Muhammad and Dambaru[7] using a modified Bernstein polynomial scheme. Abbas and Iqbal[8] proposed the cubic B-spline method for the numerical solutions of the KdV equation. In order to solve the KdV equation, Dereli and Dag[9] presented the radial basis function. Dag [10] studied the KdV equation by using the Taylor–Galerkin finite element method. The Least-squares method was used by Jacques and Arnaud[11] to find the numerical solutions of the KdV equation. Gamze and Nurcan[12] investigated the numerical solutions of the KdV equation using the Iterative splitting method. Hao [13] studied the KdV equation by using the Galerkin method. The variational iteration method was used by Mustafa[14] to solve the KdV equation. Kong [15] solved the KdV equation using a hybrid numerical scheme. Ozis and Ozer[16] explored the numerical solutions of the KdV equation using the Iterative scheme. The cosine expansion-based differential quadrature method was proposed by Saka[17] to study the KdV equation. Seadawy[18] investigated the KdV equation by using the variational approximation method. Shen[19] solved the KdV equation by the meshless method. A modified tanh–coth method was proposed by Wazzan[20] to study the KdV equation. Zhang and Ping[21] used the implicit sixth-order compact finite difference scheme for the numerical solutions of the

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KdV equation. Ozer and Kutluay[22] solved the equation by using the analytical-numerical method. Mishra [23] proposed a new quadrature method to examine the behaviour of the KdV equation.

Spline functions are widely utilized to solve initial and boundary value problems. The differential quadrature method was first introduced by Bellman[24] for solving partial differential equations. The quintic B-spline method was used by Mittal and Arora[25] for the numerical solutions of the Kuramoto–Sivashinsky equation. Zaki[26] used this method to solve the Korteweg de–Vries Burgers' equation. Dag and korkmaz[27] solved the advection-diffusion equations by quartic and quintic B-splines scheme. Saka [28] solved the regularized long wave equation by a quintic B-spline technique. Mittal and Dahiya[29] used this method for solving Fisher-Kolmogorov equations. The KdVB equation was solved by Bashan [30] using the quintic B-splines method.

The main purpose of this article is to find the numerical solutions to the kdv equation. The five-banded Thomas algorithm has been used to find the weighting coefficients. The derivative of an unknown function is expanded to obtain a system of ordinary differential equations. The system of ordinary differential equations has been solved using the SSP-RK43 scheme. This paper is divided into the following sections: In Section 2, the Quintic B-spline functions have been reproduced. The values of derivatives at the nodes have been obtained in this section. We have also explained the implementation of the method to the KdV equation in this section. In Section 3, the method has been applied to the numerical problem. The findings of the paper have been summarized in Section 4.

Quintic B-spline function

The domain c ≤ z ≤ d is discretized into a mesh of uniform length $d = z_{j+1} - z_j$, by the nodes z_j where j = 0, 1, 2, ..., M such that $c = z_1 < z_2 <, \ldots, z_{M-1} < z_M = d$. Let $S_n(z)$ be the quintic B-spline function with the nodes at points z_j. The quintic B-spline basis function at nodes, given by[31]

$$S_n(z) = \frac{1}{d^5}\begin{cases}(z-z_{n-3})^5, & z \in [z_{n-3}, z_{n-2})\\ (z-z_{n-3})^5 - 6(z-z_{n-2})^5, & z \in [z_{n-2}, z_{n-1})\\ (z-z_{n-3})^5 - 6(z-z_{n-2})^5 + 15(z-z_{n-1})^5, & z \in [z_{n-1}, z_n)\\ (z_{n+3}-z)^5 - 6(z_{n+2}-z)^5 + 15(z_{n+1}-z)^5, & z \in [z_n, z_{n+1})\\ (z_{n+3}-z)^5 - 6(z_{n+2}-z)^5, & z \in [z_{n+1}, z_{n+2})\\ (z_{n+3}-z)^5, & z \in [z_{n+2}, z_{n+3})\\ 0 & otherwise.\end{cases} \tag{2.1}$$

here $n = -1, 0, 1, \ldots, M+1, M+2$ and $\{S_{-1}, S_0, \ldots, S_{M+2}\}$ forms a basis over the region $c \le z \le d$.

Each quintic B-spline encloses six nodes, so that total of six quintic B-spline encloses one node. The nonzero values of and the first four derivatives at given node points are summarized in Table 1. With respect to the variable , the approximation to derivatives of is

$$V_z(z_j, t) = \sum_{k=1}^{M} P_{jk}^{(1)} V(z_k), \quad j = 1, 2, \ldots\ldots, M \tag{2.2}$$

here denotes the first order partial derivatives weighting coefficients with respect to z. Shu's recurrence formula[32] is used to determine the higher order derivatives.

$$P_{jk}^{(w)} = w\left[P_{jk}^{(1)} P_{jj}^{(w-1)} - \frac{P_{jk}^{(w-1)}}{z_j - z_k}\right], \quad \text{for} \quad j \neq k \tag{2.3}$$

$$j, k = 1, 2, \ldots, M; \qquad w = 2, 3, \ldots, M-1$$

$$P_{jj}^{(w)} = -\sum_{k=1, k\neq j}^{m} P_{jk}^{(w)}, \quad \text{for} \quad j = k \tag{2.4}$$

The partial derivatives weighting coefficients of order $(w-1)$ and (w) in the direction of the z-axis are indicated here by the $P_{jk}^{(w-1)}$ and $P_{jk}^{(w)}$. Substitution of each quintic B-spline function into the differential quadrature method equation (2.2) for a fixed gives

$$\frac{\partial S_n(z_j)}{\partial z} = \sum_{k=n-2}^{n+2} P_{jk}^{(1)} S_n(z_k), \tag{2.5}$$

$n = -1,\ 0, \ldots,\ M+2$ and $j = 1,\ 2, \ldots,\ M$

Table 1: and its derivatives on nodes.

z	z_{n-3}	z_{n-2}	z_{n-1}	z_n	z_{n+1}	z_{n+2}	z_{n+3}
$S_n(z)$	0	1	26	66	26	1	0
$S'_n(z)$	0	$5/d$	$50/d$	0	$-50/d$	$5/d$	0
$S''_n(z)$	0	$20/d^2$	$40/d^2$	$-120/d^2$	$40/d^2$	$20/d^2$	0
$S'''_n(z)$	0	$60/d^3$	$-120/d^3$	0	$120/d^3$	$-60/d^3$	0
$S''''_n(z)$	0	$120/d^4$	$-480/d^4$	$720/d^4$	$-480/d^4$	$120/d^4$	0

Now, we can rewrite the linear system (2.5) in the matrix notation for any z_j in the interval $[c, d]$ as given below:

$$\begin{bmatrix} S_{-1,-3} & S_{-1,-2} & S_{-1,-1} & S_{-1,0} & S_{-1,1} & & & & & \\ & S_{0,-2} & S_{0,-1} & S_{0,0} & S_{0,1} & S_{0,2} & & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & S_{M+1,M-1} & S_{M+1,M} & S_{M+1,M+1} & S_{M+1,M+2} & S_{M+1,M+3} & \\ & & & & & S_{M+2,M} & S_{M+2,M+1} & S_{M+2,M+2} & S_{M+2,M+3} & S_{M+2,M+4} \end{bmatrix} \times C_1 = \psi_1$$

here $S_{j,k}$ denotes $S_j(z_k)$, $C_1 = [P_{j,-3},\ P_{j,-2}, \dots ,\ P_{j,M+3},\ P_{j,M+4}]^T$ and $\psi_1 = [\frac{\partial S_{-1}(z_j)}{\partial z},\ \frac{\partial S_0(z_j)}{\partial z}, \dots ,\ \frac{\partial S_{M+1}(z_j)}{\partial z},\ \frac{\partial S_{M+2}(z_j)}{\partial z}]^T$

There are N+8 unknowns and N+4 equations in the linear equation system. The number of equations and unknowns is equalized by adding four more equations to the system:

$$\frac{\partial^2 S_{-1}(z_j)}{\partial z^2} = \sum_{k=-3}^{1} P_{jk}^{(1)} S_{-1}(z_k),$$

$$\frac{\partial^2 S_0(z_j)}{\partial z^2} = \sum_{k=-2}^{2} P_{jk}^{(1)} S_0(z_k),$$

$$\frac{\partial^2 S_{M+1}(z_j)}{\partial z^2} = \sum_{k=M-1}^{M+3} P_{jk}^{(1)} S_{M+1}(z_k),$$

$$\frac{\partial^2 S_{M+2}(z_j)}{\partial z^2} = \sum_{k=M}^{M+4} P_{jk}^{(1)} S_{M+2}(z_k),$$

Finally, the number of unknowns and equations are equalized and in the form of $R_1 \times C_1 = \psi_2$, here

$$R_1 = \begin{bmatrix} S_{-1,-3} & S_{-1,-2} & S_{-1,-1} & S_{-1,0} & S_{-1,1} & & & & & \\ S'_{-1,-3} & S'_{-1,-2} & S'_{-1,-1} & S'_{-1,0} & S'_{-1,1} & & & & & \\ & S_{0,-2} & S_{0,-1} & S_{0,0} & S_{0,1} & S_{0,2} & & & & \\ & S'_{0,-2} & S'_{0,-1} & S'_{0,0} & S'_{0,1} & S'_{0,2} & & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & & S_{M+1,M-1} & S_{M+1,M} & S_{M+1,M+1} & S_{M+1,M+2} & S_{M+1,M+3} & \\ & & & & S'_{M+1,M-1} & S'_{M+1,M} & S'_{M+1,M+1} & S'_{M+1,M+2} & S'_{M+1,M+3} & \\ & & & & & S_{M+2,M} & S_{M+2,M+1} & S_{M+2,M+2} & S_{M+2,M+3} & S_{M+2,M+4} \\ & & & & & S'_{M+2,M} & S'_{M+2,M+1} & S'_{M+2,M+2} & S'_{M+2,M+3} & S'_{M+2,M+4} \end{bmatrix}$$

$$C_1 = [P_{j,-3}^{(1)},\ P_{j,-2}^{(1)}, \dots ,\ P_{j,M+3}^{(1)},\ P_{j,M+4}^{(1)}]^T$$

$$\psi_2 = [\frac{\partial S_{-1}(z_j)}{\partial z},\ \frac{\partial^2 S_{-1}(z_j)}{\partial z^2},\ \frac{\partial S_0(z_j)}{\partial z},\ \frac{\partial^2 S_0(z_j)}{\partial z^2}, \dots ,\ \frac{\partial S_{M+1}(z_j)}{\partial z},\ \frac{\partial^2 S_{M+1}(z_j)}{\partial z^2},\ \frac{\partial S_{M+2}(z_j)}{\partial z},\ \frac{\partial^2 S_{M+2}(z_j)}{\partial z^2}]^T$$

We eliminated $P_{j,-3}^{(1)}, P_{j,-2}^{(1)}, P_{j,M+3}^{(1)}, P_{j,M+4}^{(1)}$ and used the derivatives and values of the quintic B-spline at the nodes to obtain a 5-band matrix of the form $R_2 \times C_2 = \psi_3$, which is a system of linear equations. here

$$R_2 = \begin{bmatrix} 37 & 82 & 21 & & & & & \\ 8 & 33 & 18 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & & 1 & 26 & 66 & 26 & 1 & \\ & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 1 & 18 & 33 & 8 \\ & & & & & 21 & 82 & 37 \end{bmatrix}$$

$\psi_3 = [\lambda_{-1},\ \lambda_0, \dots ,\ \lambda_{M+1},\ \lambda_{M+2}]$, here

$$\lambda_{-1}(z_j) = \frac{1}{30}[-5S_{-1}^{(1)}(z_j) + dS_{-1}^{(2)}(z_j) + 40S_0^{(1)}(z_j) + 8dS_0^{(2)}(z_j)],$$
$$\lambda_0(z_j) = \frac{1}{10}[5S_0^{(1)}(z_j) - dS_0^{(2)}(z_j)],$$
$$\lambda_k(z_j) = S_k^{(1)}(z_j), \quad for \quad k = 2,3,\dots,M-1,M,$$
$$\lambda_{M+1}(z_j) = \frac{1}{10}[5S_{M+1}^{(1)}(z_j) + dS_{M+1}^{(2)}(z_j)],$$
$$\lambda_{M+2}(z_j) = \frac{1}{30}[40S_{M+1}^{(1)}(z_j) - 8dS_{M+1}^{(2)}(z_j) - 5S_{M+2}^{(1)}(z_j) - dS_{M+2}^{(2)}(z_j)],$$

The penta-diagonal Thomas algorithm is used to solve the linear system $R_2 \times C_2 = \psi_3$. We obtain the approximate partial derivatives of the first order by solving and putting the values of $P_{j,-1}^{(1)}, P_{j,0}^{(1)}, \dots, P_{j,M+1}^{(1)}, P_{j,M+2}^{(1)}$ in equation (2.2). For the approximation of higher order partial derivatives and to determine $P_{jk}^{(2)}, P_{jk}^{(3)}, P_{jk}^{(4)}$, we used Shu's recurrence formula.

By substituting the derivatives approximate values with respect to z in equation (1.1), we yield the following system:

$$\frac{dV(z_j,t)}{dt} = -\varepsilon V \sum_{k=1}^{M} P_{jk}^{(1)} V_k(z_k,t) - \mu \sum_{k=1}^{M} P_{jk}^{(3)} V_k(z_k,t). \quad (2.6)$$

This system of ordinary differential equations, which provides numerical solutions at different time levels, is solved by using the SSP-RK43 scheme, which is a time stepping and stability preserving method.

Result and Discussion

The accuracy of the proposed method has been demonstrated by applying it to a single soliton equation. We have computed Maximum absolute error norm as follows:

$$L_\infty = \| V^{exact} - V_n \|_\infty \approx max|V_k^{exact} - (V_n)_k|$$

Consider the KdV equation with the exact solution stated as follows:

$$V(z,t) = 3Msech^2(Pz - Qt + L), \quad (3.1)$$

where 3M and εM indicate amplitude and velocity respectively. We have taken $P = \frac{1}{2}(\varepsilon M/\mu)^{1/2}$ and $Q = \frac{1}{2}\varepsilon M(\varepsilon M/\mu)^{1/2}$. The initial condition is obtained from the exact solution for the numerical solution of soliton at time as

$$V(z,0) = 3Msech^2(Pz + L), \quad (3.2)$$

and the boundary conditions are defined as follows:

$$V(0,t) = V(2,t) = 0 \quad for \quad t \geq 0.$$

For the numerical solution, we use $\varepsilon = 1$, $\mu = 0.000484$, $M = 0.3$, and $L = -6$. The error norm has been calculated at various time levels and is shown in Table 2. The numerical results at various time levels with various values of z have been displayed in Table 3 and represented graphically in Figures 1-3 at $t = 0.5, 1.5$, and 2.5. We can see that as time increases the single soliton moves towards the right. We found that the quintic B-spline differential quadrature method is giving better results. The ability to retain their original sizes and forms is one of the characteristics of soliton waves.

Table 2: Maximum absolute errors at various time levels

t	L_∞
0.5	4.672×10^{-4}
1.0	1.363×10^{-4}
1.5	1.882×10^{-3}
2.0	2.821×10^{-3}
2.5	2.311×10^{-3}

Table 3: Numerical solutions of single soliton at various time levels

z	$t = 0.5$	$t = 1$	$t = 1.5$	$t = 2$	$t = 2.5$
0.2	1.674×10^{-5}	2.017×10^{-5}	2.711×10^{-5}	2.815×10^{-5}	2.738×10^{-5}
0.4	6.748×10^{-5}	2.615×10^{-5}	2.064×10^{-5}	1.871×10^{-5}	1.761×10^{-5}
0.6	5.471×10^{-4}	1.702×10^{-4}	1.557×10^{-5}	8.637×10^{-6}	8.734×10^{-6}
0.8	8.935×10^{-5}	6.669×10^{-4}	7.018×10^{-4}	3.972×10^{-5}	3.152×10^{-6}
1.0	9.808×10^{-8}	4.878×10^{-5}	2.092×10^{-3}	2.102×10^{-3}	6.661×10^{-4}
1.2	3.589×10^{-8}	8.289×10^{-8}	2.069×10^{-5}	1.301×10^{-3}	1.924×10^{-3}
1.4	3.087×10^{-7}	9.376×10^{-8}	6.445×10^{-8}	7.891×10^{-6}	3.435×10^{-3}
1.6	2.193×10^{-9}	1.341×10^{-8}	6.563×10^{-8}	3.119×10^{-8}	2.619×10^{-5}
1.8	2.475×10^{-7}	1.541×10^{-8}	5.053×10^{-9}	7.571×10^{-9}	1.905×10^{-7}

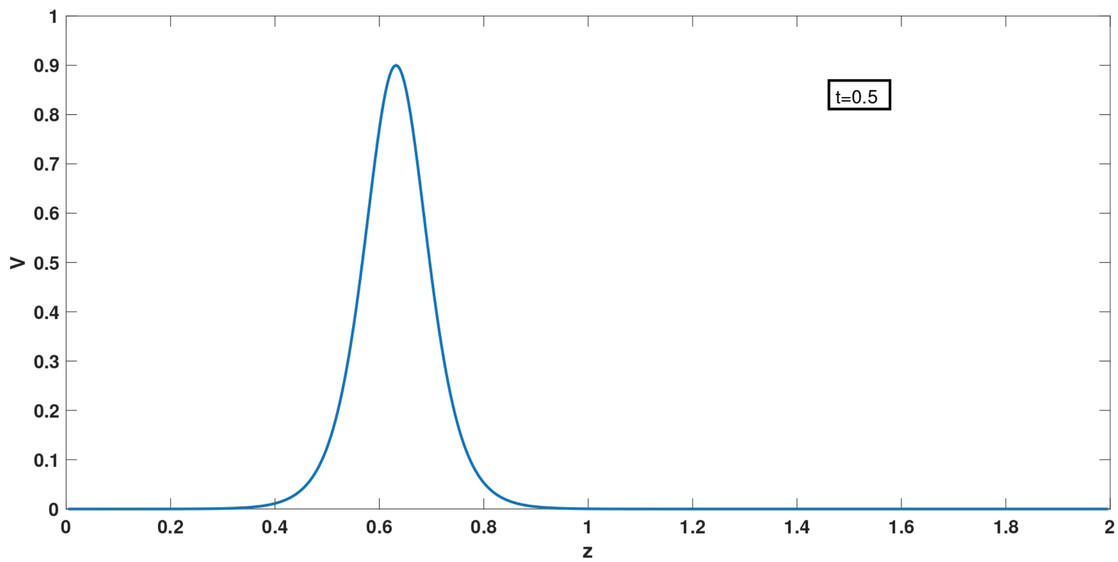

Figure 1. Simulation of single soliton at time t = 0.5

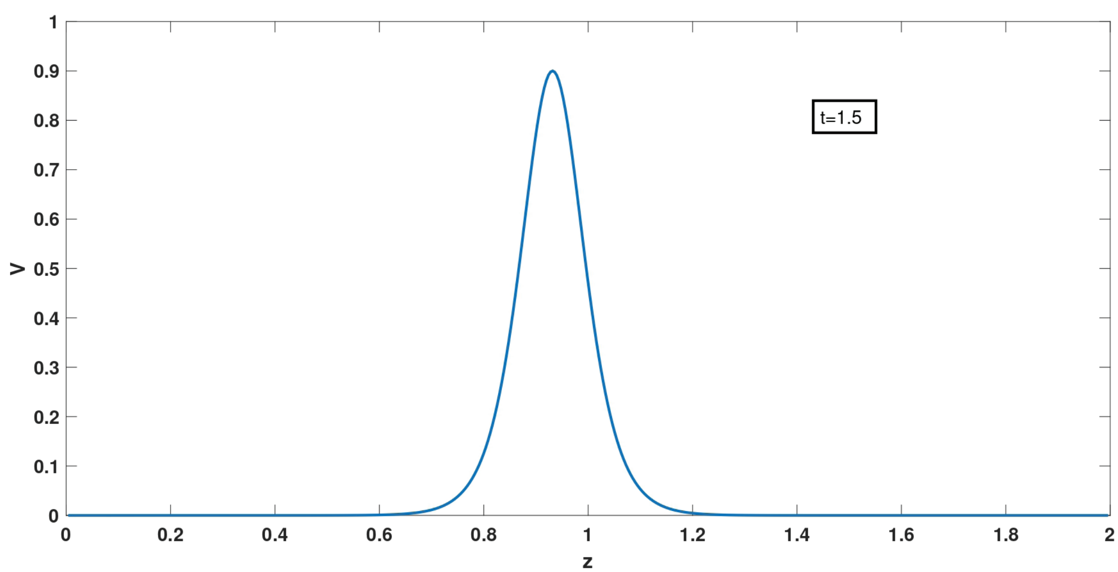

Figure 2. Simulation of single soliton at time t = 1.5

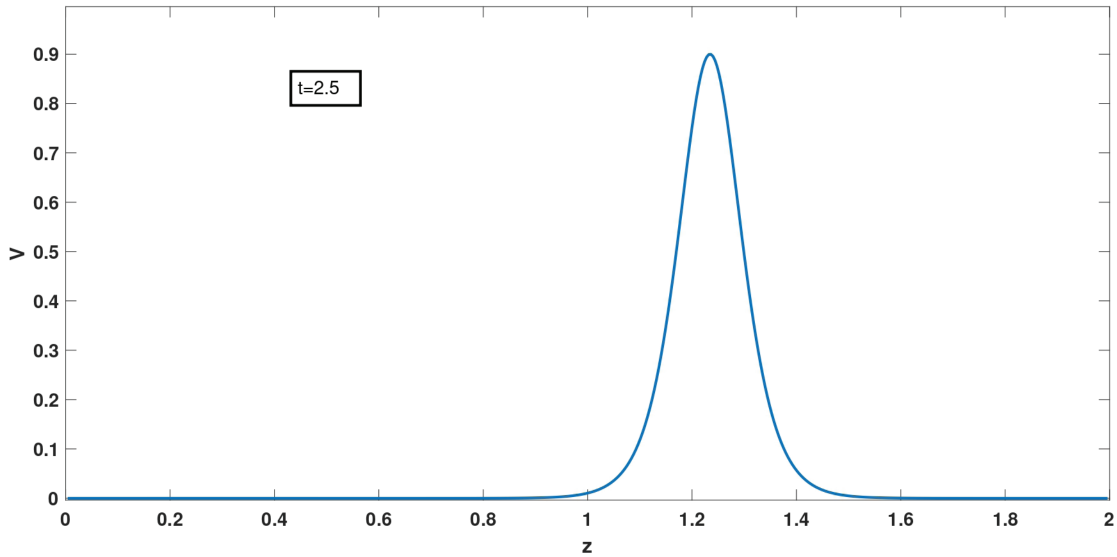

Figure 3. Simulation of single soliton at time t = 2.5

Ethical Clearance: All research has been conducted with an ethic of respect for cultures, communities, the individual/person, and independent knowledge.

Source of Funding: There is no source of funding for this research work.

Conflict of Interest: The authors declare that they do not have any conflict of interest in the current research work.

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