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Numerical Simulation of Three Dimensional Chaotic System by A Quintic B-Spline Method

Chetna Gupta[1], Rajni Rohila[2]

[1]Research Scholar, Department of Applied Sciences, the NorthCap University, Gurugram, Haryana [2]Assistant Professor, Department of applied sciences, the NorthCap University, Gurugram, Haryana

Abstract

In this work, a novel approach has been proposed in which quintic B-spline functions have been used to find numerical solutions of three dimensional chaotic dynam- ical systems. Some dynamical systems depend very sensitively on the initial conditions and exhibit interesting physical behaviour efficiently and accurately. Computed results have been illustrated in figures. The suggested method was found to be more suitable compared to Rumge Kutta fourth order method.

Keywords: *Three dimensional chaotic system, Quintic B-spline function, Differential quadrature method.*

Introduction

Chaotic theory is used to describe weather systems,[1] fluid dynamics,[2] population dy- namics[3] and is also used to model dynamical systems that are highly sensitive to initial conditions. The Lorenz system is a typical ordinary differential equation which was first investigated by Edward Lorenz. Chaotic systems are characteristically sensitive to initial conditions. Thus, as demonstrated by the butterfly effect,[4] a slight change in the initial condition could have a significant impact on the outcome.

We have analysed Lorenz's numerical solutions in the presented computational study. The system is based on the quintic B-spline method. The Lorenz system's equations are given as follows

$$\frac{dx}{dt} = \delta(v - u), \tag{1.1}$$

$$\frac{dy}{dt} = \eta u - v - uw, \tag{1.2}$$

$$\frac{dz}{dt} = uv + \theta w. \tag{1.3}$$

Here u, v and w are dynamical variables of Lorenz system and δ,η, θ are some constants. In order to find the numerical solution of Zhou's chaotic equation, Roslan et al.[5] have implemented a fourth order Runge Kutta method. For solving the numerical problems of the Lu chaotic system, Mehdi and Kareem[6] has proposed a fourth order Runge Kutta method. Chung and Freund[7] proposed an optimization method for a chaotic turbulent flow system. Allan[8] has applied a Homotopy analysis method to find a numerical solution of chaotic dynamical systems.

For the solution of non linear differential equations, the quintic B-spline method has been applied. It is used to calculate the numerical solutions of the various equations. Iqbal et al.[9] have studied a second order coupled nonlinear Schrodinger equation by using quintic B-spline method. Mittal and Dahiya[10] proposed a Kuramoto Sivashinsky equation by using quintic B-spline method. Quintic B-spline collocation method has been applied by mohammadi[11] to solve Black Scholes equation. Mirzaee and Alipour[12] proposed an n-dimensional stochastic Ito-Volterra integral equations by using quintic B-spline method. Chandrasekharan et al.[13] have studied

Corresponding Author:
Rajni Rohila
Assistant Professor, Department of applied sciences, The NorthCap University, Gurugram, Haryana
e-mail: rajnirohila@ncuindia.edu

a nonlinear modified Burgers' equation by using quintic trigonometric spline method. Quintic B-spline method has been applied by Zaki[14] to solve KdVB equation. Kaur and Joshi[15] proposed for the coupled Korteweg-de Vries equation by using quintic method of quadrature analysis based on hyperbolic B-spline. Saka et al.[16] have studied a regularized long wave equation by using quintic B-spline collocation method. Ren et al.[17] used the quintic B-spline collocation method to find a Bona–Smith family of the Boussinesq equation.

Rest of the paper has been arranged as follows: Differential quadrature method and quintic B spline functions have been discussed in Section 2. In Section 3, implementation of the function to dynamical systems is explained. Application of the method to Lorenz system and numerical solutions have been discussed in Section 4. Findings of the research work has been summarized in Section 5.

Differential quadrature method and Quintic B-splines

R.E. Bellman and his associates have developed a differential quadrature method in 1970s. It is a numerical discretization technique for obtaining accurate numerical solu- tions. For the differential quadrature method, the r–th order derivative of function *f* approximated by the formula may be used.

$$f_t^{(r)} = \sum_{m=0}^{n} a_{nm}^{(r)} f(t_m), \quad n = 0, 1, 2 \ldots N. \tag{2.1}$$

where $a_{nm}^{(r)}$ represents r-th order weighting coefficient and $f_t^{(r)}$ is r-th order derivative of f with respect to t.

Quintic B-splines: Quintic B-splines functions $Q_m(t)$ may be defined as follows

$$Q_m(t) = \frac{1}{h^5} \begin{cases} (t - t_{m-3})^5, & \mathrm{t} \in [t_{m-3}, t_{m-2}), \\ (t - t_{m-3})^5 - 6(t - t_{m-2})^5, & \mathrm{t} \in [t_{m-2}, t_{m-1}), \\ (t - t_{m-3})^5 - 6(t - t_{m-2})^5 + 15(t - t_{m-1})^5 & \mathrm{t} \in [t_{m-1}, t_m), \\ (t_{m+3} - t)^5 - 6(t_{m+2} - t)^5 + 15(t_{m+1} - t)^5 & \mathrm{t} \in [t_m, t_{m+1}), \\ (t_{m+3} - t)^5 - 6(t_{m+2} - t)^5, & \mathrm{t} \in [t_{m+1}, t_{m+2}), \\ (t_{m+3} - t)^5, & \mathrm{t} \in [t_{m+2}, t_{m+3}), \\ 0, & \text{otherwise} \end{cases} \tag{2.2}$$

Table 1.

t	t_{m-3}	t_{m-2}	t_{m-1}	t_m	t_{m+1}	t_{m+2}	t_{m+3}
$Q_m(t)$	0	1	26	66	26	1	0
$Q_m^{t}(t)$	0	$\frac{5}{h}$	$\frac{50}{h}$	0	$\frac{-50}{h}$	$\frac{-5}{h}$	0
$Q_m^{tt}(t)$	0	$\frac{20}{h^2}$	$\frac{40}{h^2}$	$\frac{-120}{h^2}$	$\frac{40}{h^2}$	$\frac{20}{h^2}$	0
$Q_m^{ttt}(t)$	0	$\frac{60}{h^3}$	$\frac{-120}{h^3}$	0	$\frac{120}{h^3}$	$\frac{-60}{h^3}$	0

Where $h = t_m - t_{m-1}$ for all m, m = −1, 0, 1....N + 1, N + 2 and $Q_{-1}, Q_0, Q_1, \ldots, Q_{N+1}, Q_{N+2}$ form a basis over a region a ≤ t ≤ b. The B-splines and its initial five derivatives may be obtained at the nodes as indicated in Table 1 by exploiting these basis functions.

A spline is a piecewise defined polynomial function that is used to approximate or in- terpolate a set of points. Splines are known for their smoothness, meaning they provide a continuous and differentiable curve. B-splines are defined by a set of basis functions. These functions are piecewise-defined polynomials that are combined to create a smooth curve.[19]

Evaluation of weighting coefficients

Evaluation of weighting coefficients

Thomas algorithm gives the weighting coefficients of $a^{(1)}_{1,-1}$, $a^{(1)}_{1,0}$, $a^{(1)}_{1,1}$, $a^{(1)}_{1,2}$,...,$a^{(1)}_{1,N}$, $a^{(1)}_{1,N+1}$, $a^{(1)}_{1,N+2}$. In the same way, we can also find out the weighting coefficients for $m = -1$, 0, 1, 2,...,N, N+1, N+2. Using these coefficients we are able to find the first order derivative.

B-splines $Q_m(t)$ are sufficiently smooth during the interval $[a, b]$, so the first order derivatives at the nodes are approximated as follows:

$$\tilde{Q}'_l(t_n) = \sum_{m=-1}^{N+2} a^{(1)}_{nm}\, \tilde{Q}_l(t_m), \quad n = 1, 2, 3,...,N \text{ and } l = 1, 2, 3,...,N.$$

Above approximation takes the following form at the first grid point,

$$\tilde{Q}'_l(t_1) = \sum_{m=-1}^{N+2} a^{(1)}_{1m}\, \tilde{Q}_l(t_m), \quad l = 1, 2, 3,...,N \text{ , which may be written as } AA_1 = b_1$$

$$\text{where } A = \begin{pmatrix} 37 & 82 & 21 & & & & & & \\ 8 & 33 & 18 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ .. & .. & .. & .. & .. & .. & .. & .. & .. \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{pmatrix}, \ A_1 = \begin{pmatrix} a^{(1)}_{1,-1} \\ a^{(1)}_{1,0} \\ a^{(1)}_{1,1} \\ \vdots \\ a^{(1)}_{1,N+1} \\ a^{(1)}_{1,N+2} \end{pmatrix} \text{ and } \ b_1 = \begin{pmatrix} -109/2h \\ -29/h \\ 0 \\ 50/h \\ 5/h \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Determining the weighting coefficients of $a^{(1)}_{k,m}$, $m = -1, 0,...,N+1, N+2$ for the k^{th} grid point t_k of the domain, we're going in the same way, as you can see above

$$\tilde{Q}'_l(t_k) = \sum_{m=-1}^{N+2} a^{(1)}_{km}\, \tilde{Q}_l(t_m), \quad l = 1, 2, 3,....N$$

$$\begin{pmatrix} 37 & 82 & 21 & & & & & & \\ 8 & 33 & 18 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ .. & .. & .. & .. & .. & .. & .. & .. & .. \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{pmatrix} \times \begin{pmatrix} a^{(1)}_{k,-1} \\ a^{(1)}_{k,0} \\ a^{(1)}_{k,1} \\ \vdots \\ a^{(1)}_{k,N+1} \\ a^{(1)}_{k,N+2} \end{pmatrix} = \begin{pmatrix} 0 \\ . \\ -5/h \\ -50/h \\ 0 \\ 50/h \\ 5/h \\ 0 \\ . \\ 0 \end{pmatrix}$$

Determining the weighting coefficients of $a^{(1)}_{N,m}$, $m = -1, 0,...,N+1, N+2$ for the last grid point of a given domain t_N,

$$\tilde{Q}'_l(t_N) = \sum_{m=-1}^{N+2} a^{(1)}_{Nm}\, \tilde{Q}_l(t_m)\ , \quad l = 1, 2, 3,...,N$$

$$\begin{pmatrix} 37 & 82 & 21 & & & & & & \\ 8 & 33 & 18 & 1 & & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ .. & .. & .. & .. & .. & .. & .. & .. & .. \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & & 1 & 18 & 33 & 8 \\ & & & & & & 21 & 82 & 37 \end{pmatrix} \times \begin{pmatrix} a^{(1)}_{N,-1} \\ a^{(1)}_{N,0} \\ a^{(1)}_{N,1} \\ \vdots \\ a^{(1)}_{N,N+1} \\ a^{(1)}_{N,N+2} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ -5/h \\ -50/h \\ 0 \\ 29/h \\ 109/2h \end{pmatrix}$$

We can calculate these coefficients by using 5-banded Thomas algorithm. The Thomas Algorithm which is also known as the tridiagonal matrix algorithm is an efficient method for solving linear system of equations where the coefficient matrix is tridiagonal. The 5- banded Thomas Algorithm[20] is an extension of this approach for pentadiagonal matrices and it works efficiently for systems with five non-zero diagonals.

Numerical Implementation of Differential Quadrature Method to the

Dynamical Chaotic Systems

In this work, we are considering three dimensional chaotic dynamical systems with unknowns u(t), v(t) and w(t). We have discretized their first order derivatives in knots u_m using the differential quadrature method:

$$\dot{u_n} \approxeq \sum_{m=1}^{N+2} a^{(1)}_{nm} u_m, \quad n = 1, 2, ..., N, \tag{3.1}$$

$$\dot{v_n} \approxeq \sum_{m=1}^{N+2} a^{(1)}_{nm} v_m, \quad n = 1, 2, ..., N, \tag{3.2}$$

$$\dot{w_n} \approxeq \sum_{m=1}^{N+2} a^{(1)}_{nm} w_m, \quad n = 1, 2, ..., N. \tag{3.3}$$

where the original conditions were set out as follows

$u(0) = u_0$, $v(0) = v_0$, $w(0) = w_0$ (3.4)

The Lorenz system:

Using the differential quadrature from (1.1) to (1.3), we obtain

$$\sum_{m=1}^{N} a^{(1)}_{nm} u_m + a^{(1)}_{n0} u_0 = \delta(v_n - u_n), \quad n = 1, 2, ..., N, \tag{3.5}$$

$$\sum_{m=1}^{N} a^{(1)}_{nm} v_m + a^{(1)}_{n0} v_0 = \eta u_n - v_n - x_n w_n, \quad n = 1, 2, ..., N, \tag{3.6}$$

$$\sum_{m=1}^{N} a^{(1)}_{nm} w_m + a^{(1)}_{n0} w_0 = u_n v_n + \theta w_n, \quad n = 1, 2, ..., N, \tag{3.7}$$

These are nonlinear algebraic equations system which has been solved by Newton's iter- ative method.[21]

4. Numerical Experiments and Results

For the Lorenz chaotic system, we used the quintic B-spline differential quadrature method.

The Lorenz System: The Lorenz system is a set of three coupled ordinary differential equations that exhibit chaotic behavior as well as non-chaotic behavior. For chaotic behavior, we have taken a specific set of parameter values (δ = 10, η = 28, θ = − [8]) and

initial conditions (u0 = −15.8, v0 = −17.48, w0 = 35.64) in the time domain 0 ≤ t ≤ 2.5.

n is taken 10 and Δt = 0.05. For non-chaotic behavior, we have taken parameter values (δ = 10, η = 23.5, θ = − [8]) and the rest of the parameters have been set as described earlier. The numerical values of u, v and w are plotted against t in figures 1-3. Figures 4-6 shows phase portraits for the non-chaotic Lorenz system. We used η = 28 to simulate a chaotic Lorenz system, and the corresponding solutions were shown in Tables 7-12. The

Lorenzs butterfly effect is shown in Figures 10-12. The method of quintic B-spline is very precise in capturing the system's biological behaviour, we see. Table 2 presents a comparison of the difference in error rates. Compared to the RK4 method, the differential quadrature method produces better results.

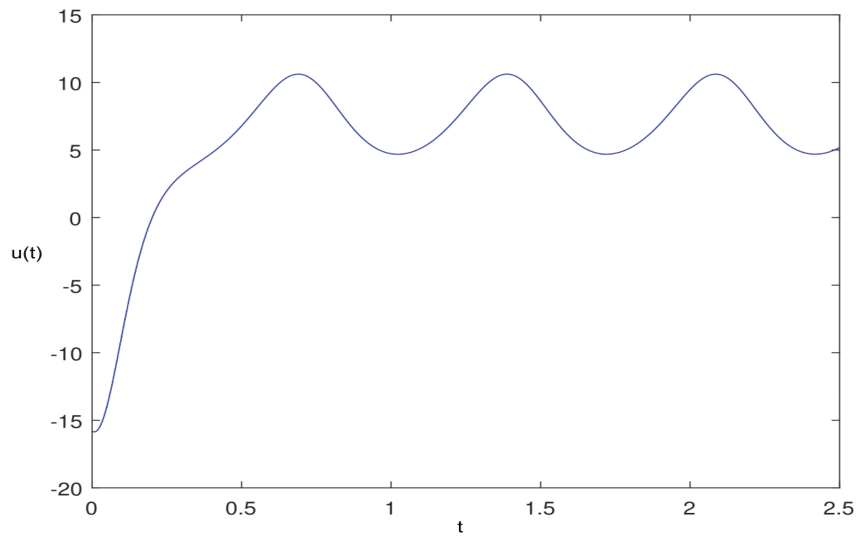

Figure 1. For the Lorenz's non-chaotic system (η = 23.5), plot of solution u.

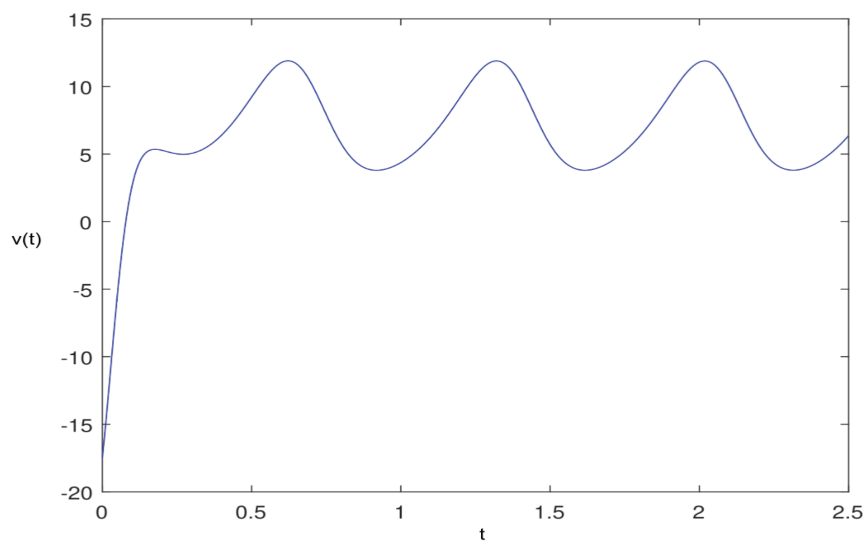

Figure 2. For the Lorenz's non-chaotic system (η = 23.5), plot of solution v.

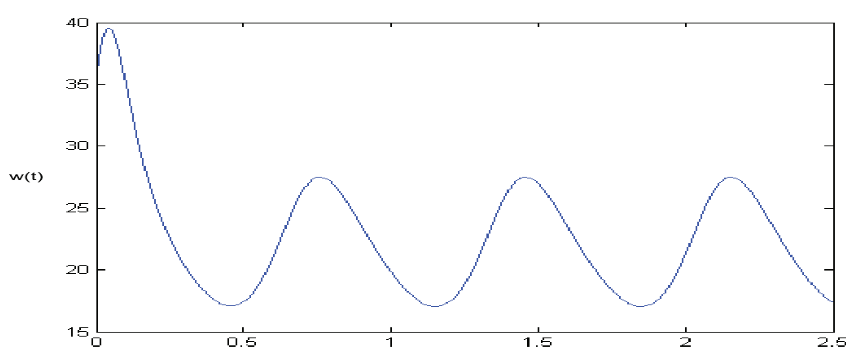

Figure 3. For the Lorenz's non-chaotic system (η = 23.5), plot of solution w.

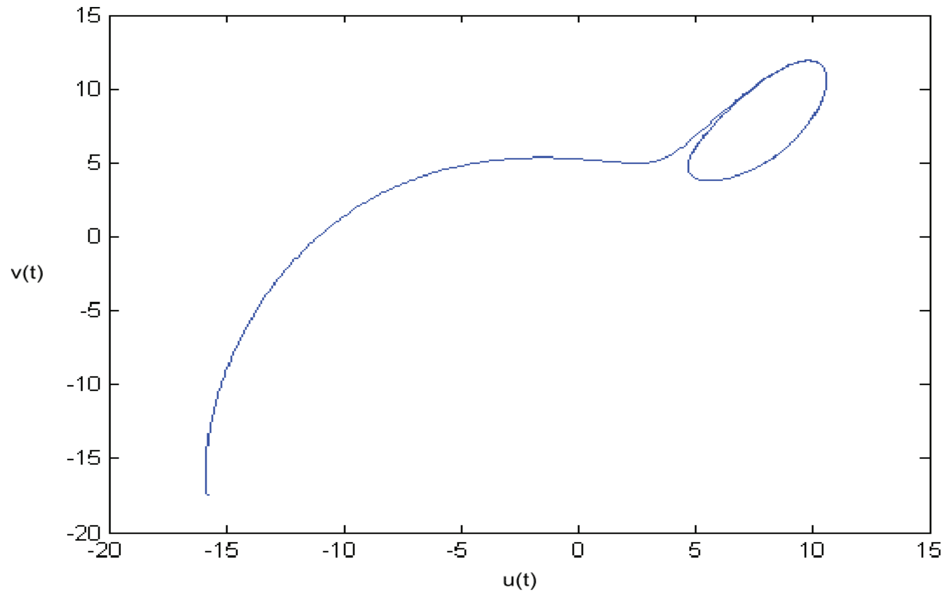

Figure 4. Phase Portrait for the non-chaotic Lorenz system (η = 23.5).

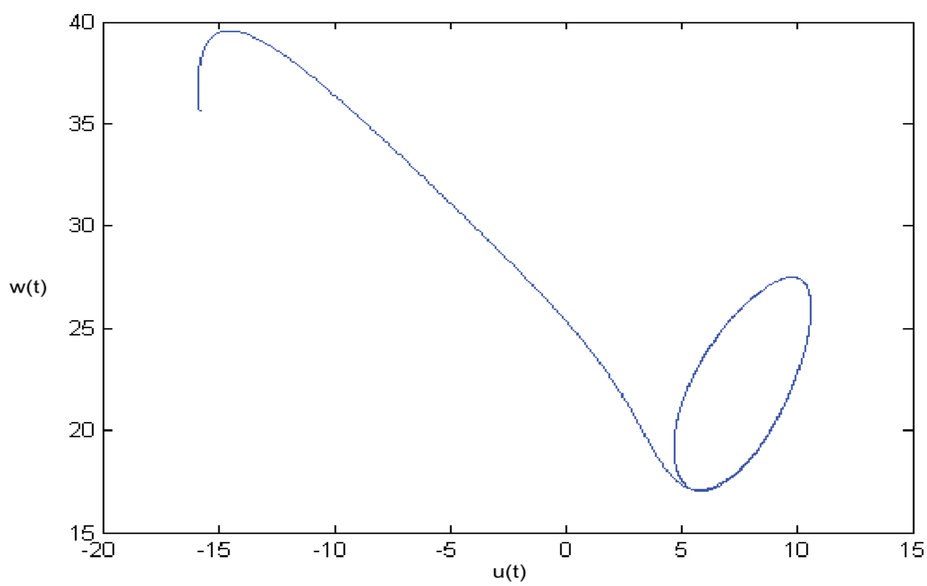

Figure 5. Phase Portrait for the non-chaotic Lorenz system (η = 23.5).

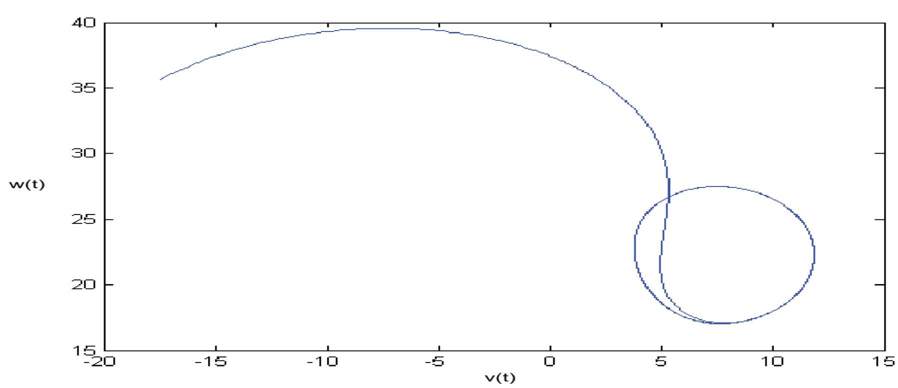

Figure 6. Phase Portrait for the non-chaotic Lorenz system (η = 23.5).

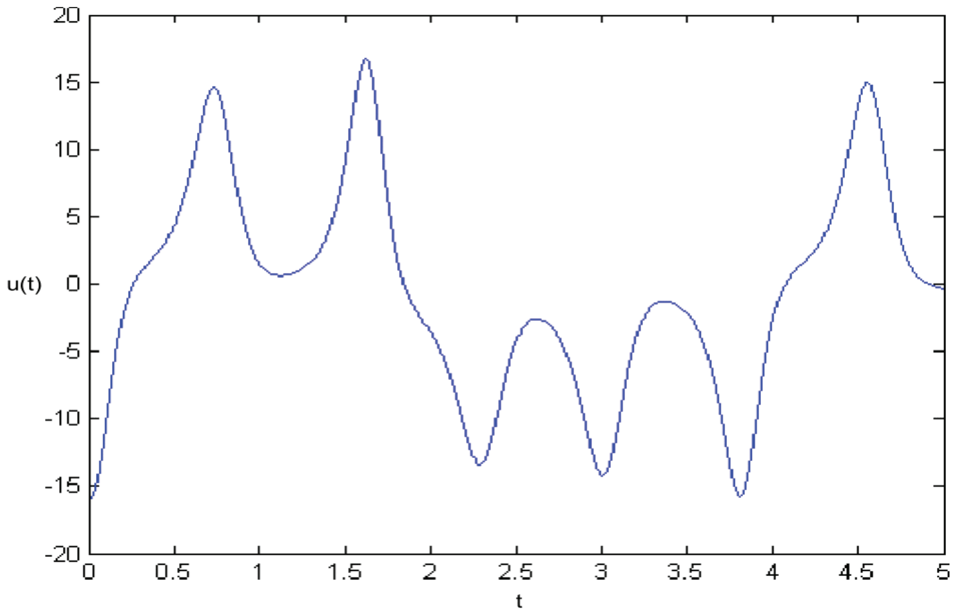

Figure 7. For the chaotic Lorenz system (η = 28), plot of solution u.

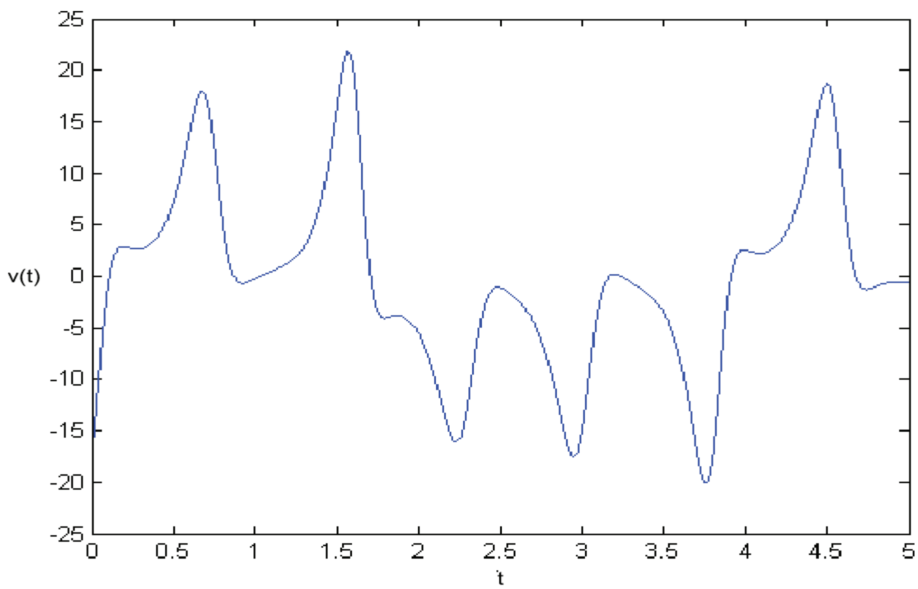

Figure 8. For the chaotic Lorenz system (η = 28), plot of solution v.

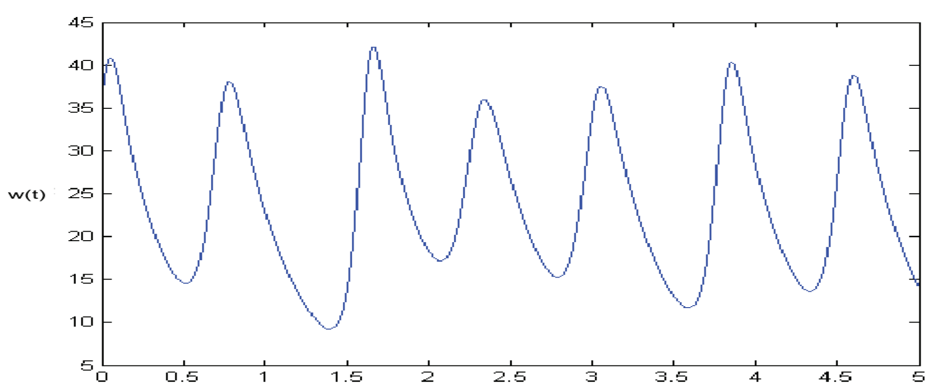

Figure 9. For the chaotic Lorenz system (η = 28), plot of solution w.

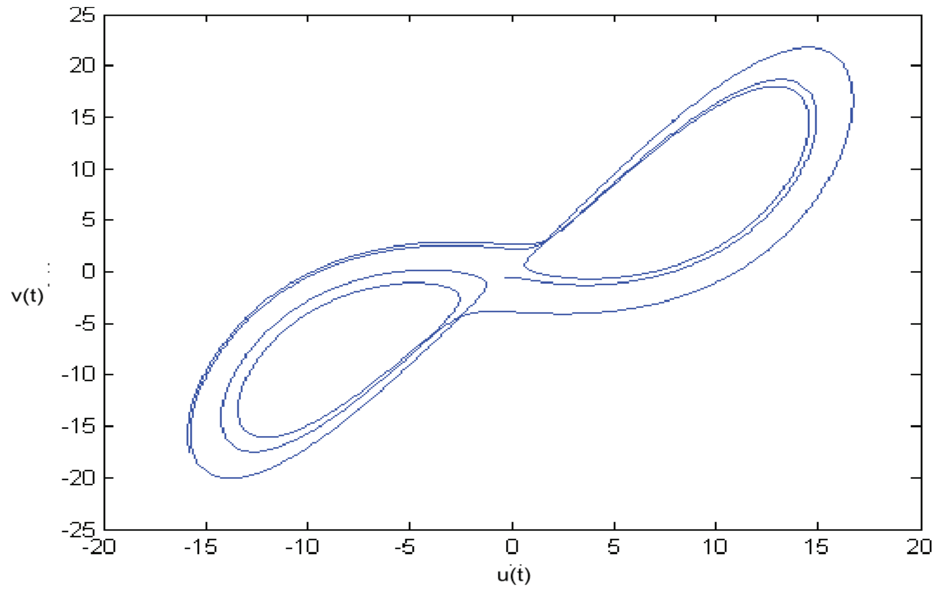

Figure 10. Phase Portrait of Lorenz's chaotic system (η = 28).

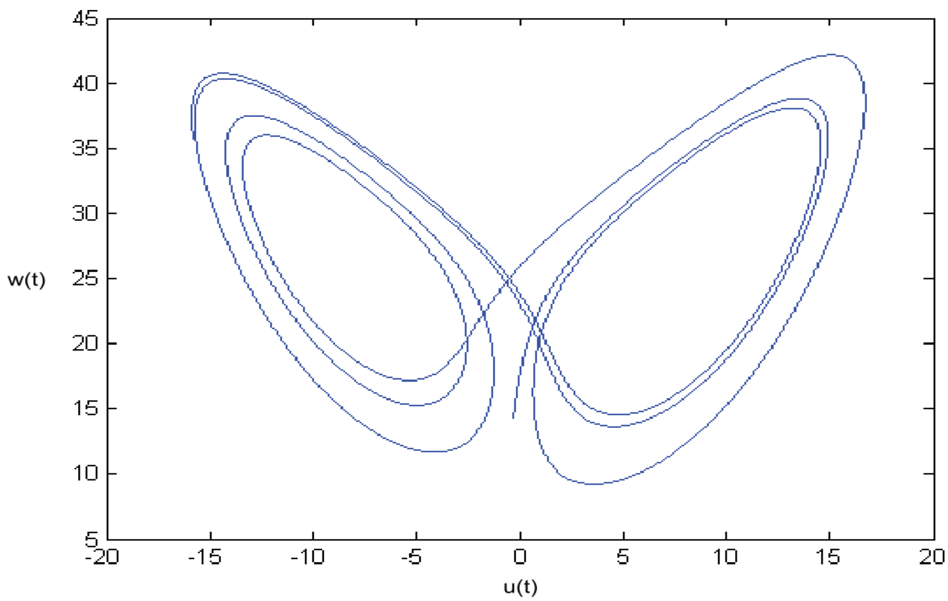

Figure 11. Phase Portrait of Lorenz's chaotic system (η = 28).

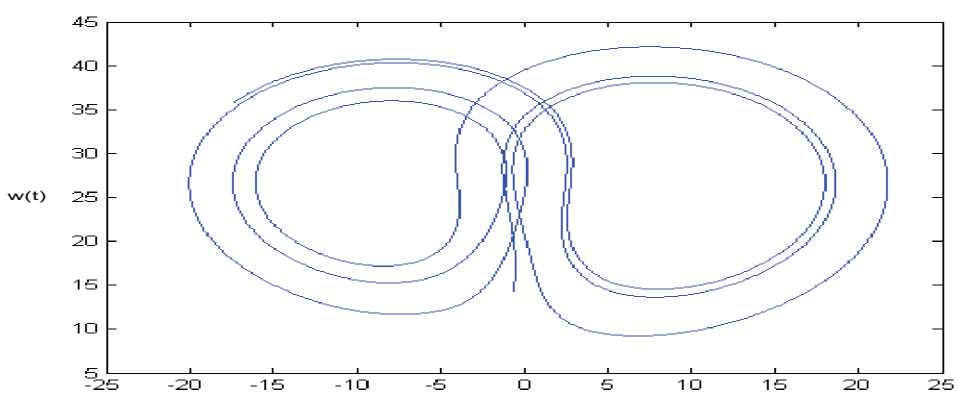

Figure 12. Phase Portrait of Lorenz's chaotic system (η = 28).

Table 2.

t	DQM Method	RK-4 Method
0.50	1.82×10^{-4}	1.09×10^{-2}
1.00	1.80×10^{-4}	2.67×10^{-2}
1.50	5.23×10^{-5}	6.30×10^{-2}
2.00	2.30×10^{-6}	1.70×10^{-2}

A comparison of the Lorenz system's relative errors

Conclusion

In this work, Lorenz chaotic dynamical systems has been studied by using quintic B-spline functions. Accurate numerical solutions have been obtained by using quintic B-splines method. Quintic B-spline method effectively handles irregular geometries, a common challenge in chaotic system simulations. Quintic B-spline method produces ac- curate and efficient numerical solutions for three-dimensional chaotic systems. Computed solutions have been illustrated graphically. We have shown that the quintic B-spline method is more advantageous than the RK-4 method. Many biological models may be solved by the quintic B-spline method.

Ethical Clearance: All research has been conducted with an ethic of respect for cul- tures, communities, the individual/person, and independent knowledge.

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